The Casimir effect and the physical vacuum

Lectures given at the intensive course “Advances in Strong-Field Electrodynamics”

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Outline

Recommended reading

1. K.A. Milton: *The Casimir Effect*, World Scientific, 2001.
2. J. Phys. **A41** No. 16, *Special Issue: Proceedings of QFEXT07*, 2008.
3. M. Bordag, U. Mohideen and V.M. Mostepanenko: *New Developments in the Casimir Effect*, Phys.Rept. **353**: 1-205, 2001. [quant-ph/0106045]
4. G.L. Klimchitskaya, U. Mohideen and V.M. Mostepanenko: *The Casimir force between real materials: experiment and theory*, Rev. Mod. Phys. **81**:1827-1885, 2009. [arXiv:0902.4022]
5. I. Brevik, J.S. Høye: *Temperature Dependence of the Casimir Force*, Eur. J. Phys. **35**: 015012, 2014. [arXiv:1312.5174]

Outline

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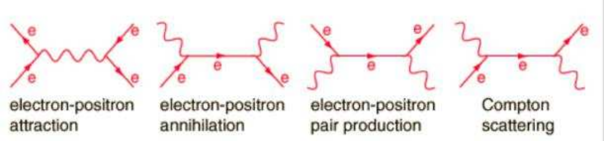
Relativistic quantum electrodynamics (QED)



1948: Feynman, Schwinger, Tomonaga (Nobel prize: 1965)

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\left(i\gamma^{\mu}(\partial_{\mu} + ieA_{\mu}) - m\right)\psi \qquad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$

Theory of the photon and the electron/positron field
(Origins: Dirac, Pauli, Weisskopf, Jordan; 1927-)

Experimental confirmation of QED

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \quad \text{fine structure constant}$$

e^- anomalous magnetic moment :	$1/\alpha = 137.035999710(96)$
Nuclear recoil:	$1/\alpha = 137.03599878(91)$
Hyperfine splitting in muonium:	$1/\alpha = 137.035994(18)$
Lamb shift:	$1/\alpha = 137.0368(7)$
Quantum Hall effect:	$1/\alpha = 137.0359979(32)$

QED: „quod erat demonstrandum"
– the most precisely validated physical theory!

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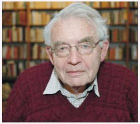
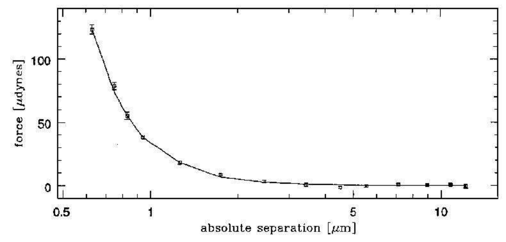
Attractive force between two perfect conductor plane in vacuum (Casimir, **1948**)



$$\frac{F}{A} = -\frac{\hbar c \pi^2}{240 a^4}$$

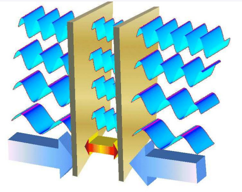
A macroscopic prediction of QED:

1 μm distance: 8.169×10^{-3} Pa



Lamoreaux, 1996: experimental verification within 5%

Naive derivation: from vacuum energy I



Scalar field with Dirichlet BC (units: $\hbar = 1 = c$)

$$\phi(z=0) = \phi(z=a) = 0$$

$$\mathcal{E} = \frac{1}{2}\sum \hbar\omega = \frac{1}{2}\sum_{n=1}^{\infty}\int \frac{d^2k}{(2\pi)^2}\sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2}$$

This is divergent, but we can use dimensional regularization. Using

$$\int_0^\infty \frac{dt}{t} t^{-n} e^{-zt} = \Gamma(-n) z^n \qquad \int d^d k\, e^{-tk^2} = \left(\frac{\pi}{t}\right)^{d/2}$$

we can write

$$\mathcal{E} = \frac{1}{2}\sum_n \int \frac{d^d k}{(2\pi)^d}\int_0^\infty \frac{dt}{t} t^{-1/2} e^{-t(k^2+n^2\pi^2/a^2)} \frac{1}{\Gamma(-1/2)}$$

$$= -\frac{1}{4\sqrt{\pi}}\frac{1}{(4\pi)^{d/2}}\sum_n \int_0^\infty \frac{dt}{t} t^{-1/2-d/2} e^{-tn^2\pi^2/a^2}$$

Naive derivation: from vacuum energy II

$$\mathcal{E} = -\frac{1}{4\sqrt{\pi}}\frac{1}{(4\pi)^{d/2}}\sum_n \int_0^\infty \frac{dt}{t} t^{-1/2-d/2} e^{-tn^2\pi^2/a^2}$$

$$= -\frac{1}{4\sqrt{\pi}}\frac{1}{(4\pi)^{d/2}}\left(\frac{\pi}{a}\right)^{1+d}\Gamma\left(-\frac{d+1}{2}\right)\sum_n n^{d+1} \qquad \text{Re } d < -1$$

$$= -\frac{1}{4\sqrt{\pi}}\frac{1}{(4\pi)^{d/2}}\left(\frac{\pi}{a}\right)^{1+d}\Gamma\left(-\frac{d+1}{2}\right)\zeta(-d-1) \qquad \text{Re } d < -2$$

$$= \infty \cdot 0 \qquad \text{for } d \text{ positive odd integer}$$

Physical: $d \in \mathbb{N} \to$ analytic continuation is needed!

$$\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-z/2} = \Gamma\left(\frac{1-z}{2}\right)\zeta(1-z)\pi^{-(1-z)/2}$$

$$\mathcal{E} = -\frac{1}{2^{d+2}\pi^{d/2+1}}\frac{1}{a^{d+1}}\Gamma\left(1+\frac{d}{2}\right)\zeta(2+d) \underset{d=3}{\to} -\frac{\pi^2}{1440}\frac{1}{a^3}$$

Pressure: $\mathcal{F} = -\frac{\partial \mathcal{E}}{\partial a} = -\frac{\pi^2}{480}\frac{1}{a^4}$ EM: $2\times$

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A physical derivation: from momentum flow I

Energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \phi(x) \partial_\nu \phi(x) - \eta_{\mu\nu} \mathscr{L}(x)$$

$$\mathscr{L}(x) = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x)$$

Left plate at $z = 0$: what we want is

$$\mathscr{F} = \langle T_{zz} \rangle_{z>0} - \langle T_{zz} \rangle_{z<0}$$

How do we compute? From QFT

$$\langle T\phi(x)\phi(x') \rangle = -iG(x, x')$$

Now

$$-\partial^2 G(x, x') = \delta(x - x')$$

$$G(x, x') = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} g(z, z'|\vec{k}, \omega)$$

$$-\left(\frac{\partial^2}{\partial z^2} - \lambda^2 \right) g(z, z') = \delta(z - z') \qquad \lambda^2 = \omega^2 - k^2$$

$$g(0, z') = g(a, z') = 0$$

A physical derivation: from momentum flow II

Internal contribution

$$g_{\text{int}}(z,z') = -\frac{1}{\lambda \sin\lambda a}\sin\lambda z_< \sin\lambda(z_> - a)$$

$$\Downarrow$$

$$t_{zz}^{\text{int}} = \frac{1}{2i}\partial_z\partial_{z'} g_{\text{int}}(z,z')|_{z\to z'=0} = \frac{i}{2}\lambda\cot\lambda a$$

so

$$\mathscr{F}_{\text{int}} = \int \frac{d^dk}{(2\pi)^d}\int\frac{d\omega}{2\pi}\frac{i}{2}\lambda\cot\lambda a$$

$$= -\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\int\frac{d\zeta}{2\pi}\kappa\coth\kappa a \quad \text{divergent!}$$

with $\omega \to i\zeta \quad \lambda \to i\kappa = i\sqrt{k^2+\zeta^2}$

Outer contribution

$$g_{\text{out}}(z,z') = \frac{1}{\lambda}\sin\lambda z_< e^{ikz_>}$$

$$t_{zz}^{\text{out}} = \frac{1}{2i}\partial_z\partial_{z'} g_{\text{out}}(z,z')|_{z\to z'=0} = \frac{1}{2}\lambda$$

A physical derivation: from momentum flow III

$$\mathscr{F} = -\frac{1}{2}\int \frac{d^d\vec{k}}{(2\pi)^d}\int \frac{d\zeta}{2\pi}\kappa(\coth\kappa a - 1) = -\Omega_{d+1}\int_0^\infty \frac{\kappa^d d\kappa}{(2\pi)^{d+1}}\frac{\kappa}{e^{2\kappa a}-1}$$

Angular integral

$$\int d^d x\, e^{-\vec{x}^2} = \left(\int d\xi\, e^{-\xi^2}\right)^d = \pi^{d/2}$$

$$= \Omega_d \int x^{d-1}e^{-x^2}dx = \Omega_d \frac{\Gamma(d/2)}{2} \quad \Rightarrow \quad \Omega_d = \frac{2\pi^{d/2}}{\Gamma[d/2]}$$

Use

$$\Gamma(2z) = \frac{2^{2z-1/2}}{\sqrt{2\pi}}\Gamma(z)\Gamma(z+1/2) \qquad \Gamma(s)\zeta(s) = \int_0^\infty dy\, \frac{y^{s-1}}{e^y - 1}$$

to get

$$\mathscr{F} = -(d+1)2^{-d-2}\pi^{-d/2-1}\frac{\Gamma(1+d/2)\zeta(d+2)}{a^{d+2}} = -\frac{\partial}{\partial a}\mathscr{E}(a)$$

$$\text{with } \mathscr{E}(a) = -\frac{1}{2^{d+2}\pi^{d/2+1}}\frac{1}{a^{d+1}}\Gamma\left(1+\frac{d}{2}\right)\zeta(2+d)$$

Outline

Massive scalar field

$$\mathscr{L}(x) = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - \frac{1}{2}m^2\phi^2$$
$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

$$-(\partial^2 + m^2)G(x,x') = \delta(x - x')$$
$$G(x,x') = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} g(z,z'|\vec{k},\omega)$$
$$-\left(\frac{\partial^2}{\partial z^2} - \lambda^2\right) g(z,z') = \delta(z - z') \qquad \lambda^2 = \omega^2 - k^2 - m^2$$
$$g(0,z') = g(a,z') = 0$$

$$\mathscr{F} = -\Omega_{d+1}\int_0^\infty \frac{\kappa^d d\kappa}{(2\pi)^{d+1}} \frac{\sqrt{\kappa^2 + m^2}}{e^{2a\sqrt{\kappa^2+m^2}} - 1}$$

Massive scalar field II; EM field; fermions

$$\mathscr{E} = \frac{1}{a^{d+1}} \frac{1}{2^{d+1}\pi^{(d+1)/2}\Gamma(\frac{d+1}{2})} \int_0^\infty dt\, t^d \log\left(1 - e^{2\sqrt{t^2+m^2a^2}}\right)$$

$$= -2\left(\frac{ma}{4\pi}\right)^{d/2+1} \frac{1}{a^{d+1}} \sum_{n=1}^{\infty} \frac{1}{n^{d/2+1}} K_{d/2+1}(2n\,ma)$$

$$K_n(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}\left(1 + O(x^{-1})\right)$$

so the effect decays exponentially with ma.

For the EM field between perfectly conducting planes one needs to consider 2 independent polarizations: $2\times$ the result for scalar with Dirichlet BC.

For fermions

$$\mathscr{L} = \bar{\psi}\gamma^\mu \partial_\mu \psi$$

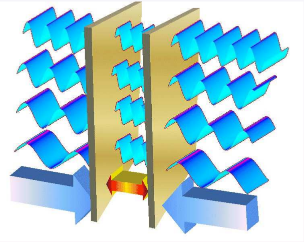
proper BC is that no conserved current flows out (bag model):

$$(1 + \vec{n}\cdot\vec{\gamma})\psi|_S = 0$$

Result for planar BC: 7/4 of the scalar force.

Outline

A popular myth: mysterious force between ships at sea



Popular myth: ships attract at heavy swell due to smaller wave pressure in between.

The two situations were messed up: Caussée claimed attraction in calm sea (below), not in a swell (above)!

Nature, doi:10.1038/news060501-7

P. C. Caussée:
The Album of the Mariner (1836)

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Temperature dependence

Matsubara formalism

$$Z = \text{Tr}\; e^{-\beta H} \qquad \beta = \frac{1}{T}$$

$$\langle \phi_2(\vec{x}) | e^{-i(t_2-t_1)H} | \phi_1(\vec{x}) \rangle = \int_{\phi(\vec{x},t_1)=\phi_1(\vec{x})}^{\phi(\vec{x},t_2)=\phi_2(\vec{x})} [d\phi]\, e^{i\int_{t_1}^{t_2} dt \int d^dx \mathscr{L}}$$

$$\Downarrow \tau = it \qquad \mathscr{L}_E = -\mathscr{L}|_{t\to -i\tau}$$

$$Z = \int_{\phi(\vec{x},\beta)=\phi(\vec{x},0)} [d\phi]\, e^{-\int_0^\beta d\tau \int d^dx \mathscr{L}_E}$$

Due to PBC in τ, the Euclidean frequencies are quantized

$$\zeta \to \zeta_n = \frac{2\pi n}{\beta} \qquad \text{fermions: APBC } \zeta_n = \frac{\pi(2n+1)}{\beta}$$

$$\int \frac{d\zeta}{2\pi} \to \frac{1}{\beta}\sum_n$$

$$\mathscr{F}_T = -\frac{1}{\beta}\int \frac{d^d\vec{k}}{(2\pi)^d} \sum_n \frac{\kappa_n}{e^{2\kappa_n a}-1} \qquad \kappa_n = \sqrt{k^2 + \left(\frac{2\pi n}{\beta}\right)^2}$$

High-temperature limit is classical

$T \to \infty$: only $n = 0$ term

$$\mathscr{F}_T = -T \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{k}{e^{2ka} - 1} = -T \frac{d}{(2\sqrt{\pi} a)^{d+1}} \Gamma\left(\frac{d+1}{2}\right) \zeta(d+1)$$

Classical free energy

$$F = -T \log Z = T \sum_{\vec{p}} \log(1 - e^{-\beta|\vec{p}|})$$

$$= TV \int \frac{d^d \vec{k}}{(2\pi)^{d+1}} \frac{\pi}{a} \sum_{n=-\infty}^{\infty} \log\left(1 - e^{-\beta\sqrt{\vec{k}^2 + n^2\pi^2/a^2}}\right)$$

For $T \to \infty$ expand exponential and use $\log \xi = \frac{d}{ds} \xi^s \Big|_{s=0}$

$$F \sim TV \frac{1}{2a} \frac{d}{ds} \int \frac{d^d \vec{k}}{(2\pi)^{d+1}} \sum_{n=-\infty}^{\infty} \frac{1}{2} \beta^{2s} \left(\frac{n^2\pi^2}{a^2} + k^2\right)^s \Bigg|_{s=0}$$

$$= -TV \frac{1}{(2\sqrt{\pi} a)^{d+1}} \Gamma\left(\frac{d+1}{2}\right) \zeta(d+1)$$

High-temperature limit is classical

$$F \sim TV \frac{1}{2a} \frac{d}{ds} \int \frac{d^d \vec{k}}{(2\pi)^{d+1}} \sum_{n=-\infty}^{\infty} \frac{1}{2} \beta^{2s} \left(\frac{n^2 \pi^2}{a^2} + k^2 \right)^s \Bigg|_{s=0}$$

Now do the momentum integral, perform the summation using ζ-function and use

$$\frac{d}{ds} \frac{1}{\Gamma(-s)} \Bigg|_{s=0} = -1$$

So the free energy is

$$F = -TV \frac{1}{(2\sqrt{\pi} a)^{d+1}} \Gamma\left(\frac{d+1}{2}\right) \zeta(d+1)$$

Now the pressure is

$$\mathscr{F} = -\frac{\partial F}{\partial V} \qquad V = Aa \Rightarrow \frac{\partial}{\partial V} = \frac{1}{A} \frac{\partial}{\partial a}$$

and this gives the same result

$$\mathscr{F}_T = -T \frac{d}{(2\sqrt{\pi} a)^{d+1}} \Gamma\left(\frac{d+1}{2}\right) \zeta(d+1)$$

Low-temperature limit

This is much more complicated: the result is not analytic in T. The leading correction is

$$\mathscr{F} \approx -(d+1)2^{-d-2}\pi^{-d/2-1}\frac{\Gamma(1+d/2)\zeta(d+2)}{a^{d+2}} \times \left(1+\frac{1}{d+1}\left(\frac{2a}{\beta}\right)^{d+2}\right)$$

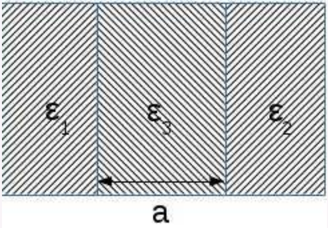
but there are also corrections of the form

$$\left(\frac{a}{\beta}\right)^{\cdots} e^{-\dots\pi\beta/a}$$

For details cf. Milton's book.

Outline

Lifschitz theory for dielectrics
in planar geometry



$$\mathscr{F}^{T=0} = -\frac{1}{16\pi^3}\int_0^\infty d\zeta \int d^2\vec{k}\, 2\kappa_3 \left(\frac{1}{d} + \frac{1}{d'}\right)$$

$$\text{TE: } d = \frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1}\frac{\kappa_3+\kappa_2}{\kappa_3-\kappa_2}e^{2\kappa_3 a} - 1 \qquad \text{TM: } d' = d(\kappa \to \kappa/\varepsilon)$$

$$\kappa^2 = k^2 + \varepsilon\zeta^2 \qquad (\zeta = i\omega)$$

Finite temperature:

$$\zeta \to \zeta_n = \frac{2\pi n}{\beta}$$

$$\int_0^\infty \frac{d\zeta}{2\pi} \to \frac{1}{\beta}\sum_{n=0}^{\infty}{}' \qquad (n = 0 \text{ with half weight})$$

One can rewrite the force ($\varepsilon_1 = \varepsilon_2 = \varepsilon$ and $\varepsilon_3 = 1$)

$$\mathscr{F}^T = -\frac{T}{\pi} \sum_{n=0}^{\infty}{}' \int_{\zeta_n}^{\infty} q^2 dq \left[\underbrace{\frac{A_n e^{-2qa}}{1 - A_n e^{-2qa}}}_{\text{TM mode}} + \underbrace{\frac{B_n e^{-2qa}}{1 - B_n e^{-2qa}}}_{\text{TE mode}} \right]$$

$$\zeta_n = 2\pi n T$$

$$A_n = \left(\frac{\varepsilon p - s}{\varepsilon p + s} \right)^2 \qquad B_n = \left(\frac{p - s}{p + s} \right)^2$$

$$s^2 = \varepsilon - 1 + p^2 \qquad p = \frac{q}{\zeta_n}$$

Limit of ideal metal: $\varepsilon(i\zeta_n) \to \infty$. However, in the zero-frequency TE mode, the limits do not commute:

$$\text{first } \varepsilon \to \infty \text{ then } \zeta \to 0 : \quad B_0 \to 1$$

$$\text{first } \zeta \to 0 \text{ then } \varepsilon \to \infty : \quad B_0 \to 0$$

Reflectivity of metals

In terms of reflectivity

$$A_n = r^{(1)}_{TM}(i\zeta_n, \vec{k}_\perp) r^{(2)}_{TM}(i\zeta_n, \vec{k}_\perp) \qquad B_n = r^{(1)}_{TE}(i\zeta_n, \vec{k}_\perp) r^{(2)}_{TE}(i\zeta_n, \vec{k}_\perp)$$

Ideal metals $\varepsilon = \infty$

$$r_{TM}(\omega, \vec{k}_\perp) = 1 \qquad r_{TE}(\omega, \vec{k}_\perp) = -1$$

so $A_n = B_n = 1$ for all n.
For real metals $\varepsilon < \infty$

$$r_{TM}(0, \vec{k}_\perp) = 1 \qquad r_{TE}(0, \vec{k}_\perp) = 0$$

so $B_0 = 0$, and stays so in the limit $\varepsilon \to \infty$.
Casimir free energy per unit surface

$$F = \frac{T}{2\pi} \sum_{n=0}^{\infty}{}' \int_{\zeta_n}^{\infty} q dq \left[\underbrace{\log\left(1 - A_n e^{-2qa}\right)}_{\text{TM mode}} + \underbrace{\log\left(1 - B_n e^{-2qa}\right)}_{\text{TE mode}} \right]$$

$$\mathscr{F}^T = -\frac{\partial F}{\partial a}$$

Ideal metal

Ideal metal: $A_n = B_n = 1$ for all n. Then

$$\mathscr{F}^T = -\frac{\pi^2}{240a^4}\left[1 + \frac{1}{3}(2aT)^4\right] \qquad aT \ll 1$$

Casimir free energy per unit surface

$$F = -\frac{\pi^2}{720a^3}\left[1 + \underbrace{\frac{45\zeta(3)}{\pi^3}(2aT)^3}_{\text{requires special care}} - (2aT)^4\right] \qquad \zeta(3) \approx 1.2$$

Entropy

$$S = -\frac{\partial F}{\partial T} = \frac{3\zeta(3)}{2\pi}T^2 - \frac{4\pi^2 a}{45}T^3 \quad aT \ll 1$$

This is fine: $S(T \to 0) = 0$.

Drude model

$$\varepsilon(i\zeta) = 1 + \frac{\omega^2_{plasma}}{\zeta(\zeta+\nu)}$$

very good model for many metals in optical experiments for $\zeta < 2 \cdot 10^{15}\ Hz$
(e.g. gold: $\omega_p = 9.03\ eV$, $\nu = 0.0345\ eV$). Whenever

$$\lim_{\zeta \to 0} \zeta^2 \left(\varepsilon(i\zeta) - 1\right) = 0$$

the zero-frequency TE mode does not contribute, i.e. $B_0 = 0$:

$$\mathscr{F}^T = -\frac{\pi^2}{240a^4}\left[1 + \frac{1}{3}(2aT)^4\right] + \frac{T}{8\pi a^3}\zeta(3) \qquad aT \ll 1$$

$$F = -\frac{\pi^2}{720a^3}\left[1 + \frac{45\zeta(3)}{\pi^3}(2aT)^3 - (2aT)^4\right] + \frac{T}{16\pi a^2}\zeta(3)$$

$$S = \frac{3\zeta(3)}{2\pi}T^2 - \frac{4\pi^2 a}{45}T^3 - \frac{\zeta(3)}{16\pi a^2} \qquad \text{!!! violates Nernst theorem}$$

Proposed solutions

Mostepanenko, Geyer: abandon Drude model.
Low frequency $\Rightarrow$ wave-length long, field constant inside plate$\Rightarrow$ cannot exist, leads to charge separation

However: why to give up a successful description of materials, when there are other ways to avoid the problem.
E.g. if resistivity does not simply go to 0 at $T = 0$, i.e.

$$\nu(T \to 0) \neq 0$$

Additional physical effects:
1. Spatial dispersion

$$\varepsilon(\omega, \vec{k})$$

Only $\varepsilon(0,0)$ would be infinite, but that is zero measure in $\vec{k}$ space.

Proposed solutions II

2. Anomalous skin effect: mean free path of electrons becomes longer than field penetration depth near $T = 0$.
Again, no contribution from TE zero mode found.

3. Large separation: result for Casimir effect same as for large T, i.e. classical. It turns out TE modes do not contribute in this limit and

$$\mathscr{F} = -\frac{\zeta(3)\,T}{8\pi a^3} \qquad a \to \infty$$

and this precisely agrees with the Drude prediction.

Future experiments will decide which scenario is valid (possibly dependent on material).

Present experimental situation seems inconclusive to me.

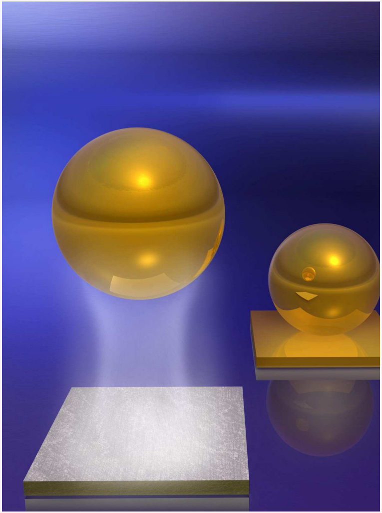
Repulsive Casimir forces

One way: measure inside fluid,
suitably chosen dielectric constant
$\Rightarrow$ Lifshitz theory predicts repulsion.
J.N. Munday, F. Capasso, and V.A. Parsegian:
Nature **457**: 170–173, 2009.

Gold sphere - gold plate, in bromobenzene:
150 pN at 20 nm separation

Other way: coat surfaces of appropriate (meta)materials
e.g. $\varepsilon_{left} = \infty$ and $\mu_{right} = \infty$
or negative refraction (cloaking)
(KK: only in limited freq. range!)

Analysis: K.A Milton et al, J. Phys. **A45** 374006, 2012. [arXiv:1202.6415]



Puzzle

What do you get if you lay an an invisibility cloak on the floor?

 $\Rightarrow$ 

A flying carpet!

Outline

Where has α gone?

QED effect: would expect appearance of fine structure constant, but it is nowhere to be found...

Actual metals: frequency-dependent dielectric constant and conductivity. Drude model:

$$\sigma(\omega) = 0 \qquad \omega^2 > \omega_{plasma}^2 = \frac{4\pi e^2 n}{m}$$

For $\omega < \omega_{plasma}$: penetration length (skin depth)

$$\delta^{-2} = \frac{2\pi\omega|\sigma|}{c^2} \qquad \sigma = \frac{ne^2}{m(\gamma_0 - i\omega)}$$

Typically $\omega \gg \gamma_0$ (damping)

$$\delta \approx \frac{c}{\sqrt{2}\omega_{plasma}}$$

Frequencies dominating Casimir effect: $c/d \Rightarrow$ perfect conductor approximation means

$$\frac{c}{d} \ll \omega_{plasma} \qquad \alpha \gg \frac{mc}{4\pi\hbar n d^2}$$

Where has α gone? II

Typically: $d < 0.5\mu m$. Copper:

$$\frac{mc}{4\pi\hbar n d^2} \approx 10^{-5} \ll \alpha \approx \frac{1}{137}$$

Casimir force is $\alpha \to \infty$ limit!!!
$\alpha \to 0$ limit:

$$a_{Bohr} = \frac{\hbar^2}{me^2} \propto \frac{1}{\alpha}$$

and so $n \propto \alpha^3 \quad \Rightarrow \quad \omega_{plasma} \propto \alpha^2$: for any fixed separation d, Casimir effect goes away.

Also $\delta \to \infty$: separation d becomes ill-defined.

For more details
cf. R.L. Jaffe: The Casimir effect and Quantum Vacuum, hep-th/0503158.

Radiative corrections: Schwinger's method

Schwinger's approach: consider the vacuum persistence amplitude in the presence of sources and boundaries

$$e^{iW[K]} = \langle 0|e^{-iHT}|0\rangle = \int D\Phi e^{i(S[\Phi]+\int K\Phi)}$$

$$W[K] = \frac{1}{2}\int dxdx' K(x)G(x,x')K(x')$$

Effective field

$$\phi(x) = \int dx' G(x,x')K(x')$$

$$K(x) = \int dx' G^{-1}(x,x')\phi(x')$$

Altering the geometry (e.g. moving boundaries adiabatically)

$$\delta W[K] = \frac{1}{2}\int dxdx' K(x)\delta G(x,x')K(x')$$

$$= -\frac{1}{2}\int dxdx' \phi(x)\delta G^{-1}(x,x')\phi(x')$$

Now

$$e^{iW[K]} = e^{\frac{1}{2}i\int dx K(x)\phi(x)} = \cdots - \frac{1}{2}\int dx dx' \phi(x)K(x)K(x')\phi(x')$$

i.e. changing boundaries is equivalent to a new two-particle source

$$\left[iK(x)K(x')\right]_{eff} = -\delta G^{-1}(x,x')$$

$$\delta W = \frac{i}{2}\int dx dx' G(x,x')\delta G^{-1}(x,x') = -\frac{i}{2}\int dx dx' \delta G(x,x') G^{-1}(x,x')$$
$$= -\frac{i}{2}\int dx dx' \delta \log G(x,x') = -\frac{i}{2}\delta \mathrm{Tr}\log G$$

so

$$E = \lim_{T\to\infty}\frac{i}{2T}\left(\mathrm{Tr}\log G - \mathrm{Tr}\log G_{ref}\right)$$

where G_{ref} is the value at some reference state (e.g. with bodies infinite distance apart).

Radiative correction for electromagnetic field

Use perturbative form of G with Π as polarization

$$G = G_0(1 + \Pi G_0 + \dots)$$

Result for parallel plates

$$\mathscr{E} = \frac{E}{A} = -\frac{\pi^2}{720a^3} + \frac{\alpha\pi^2}{2560 m_e a^4} + O(\alpha^2)$$

This is suppressed by

$$\frac{\alpha m_e^{-1}}{a}$$

and is inobservable in practice

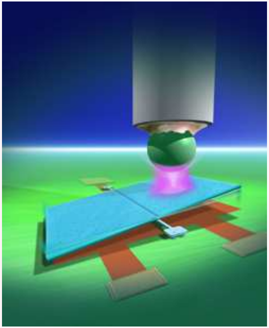
$$m_e^{-1} = \lambda_{Compton} \approx 2.43 \cdot 10^{-12} m$$
$$\alpha \approx \frac{1}{137}$$

Outline

Outline

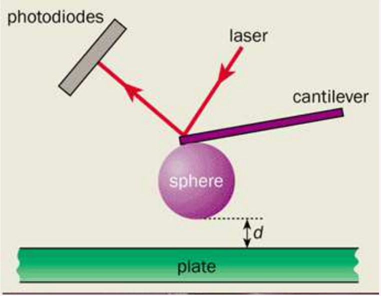
Novel measurement methods

Figure : Bell Labs



Torsion balance
(Capasso, Harvard)

Figure : Mohideen et al.



AFM (Atomic Force Microscope), sensibility in principle can be 10^{-17} N (reached: 10^{-13} N)
Si-plate: dielectric constant can be modulated by laser
(U. Mohideen et al., UC Riverside)

Proximity force approximation; special geometries

Simplest way to account for geomery dependence:

Proximity Force Theorem

Sphere and plate, $R \gg d$: every element of sphere is approximately parallel to plate

$$V(d) = \int_0^\pi 2\pi R \sin\theta R d\theta \, \mathscr{E}(d + R(1-\cos\theta)) = 2\pi R \int_{-R}^{R} dx \mathscr{E}(d + R - x)$$

$$F = -\frac{\partial V}{\partial d} = 2\pi R \int_{-R}^{R} dx \frac{d\mathscr{E}(d+R-x)}{dx}$$

$$= 2\pi R(\mathscr{E}(d) - \mathscr{E}(d+2R)) \approx 2\pi R \mathscr{E}(d)$$

Lamoreaux: 5% $\rightarrow$ Mohideen & Roy: 1%$\rightarrow$ Bell Labs 0.5%

Need to include: finite conductivity corrections, surface roughness.

Other calculations: sphere - plate, cylinder - plate, concentric spheres, coaxial cylinders.

(K.A.Milton: The Casimir effect, World Scientific, 2001.)

Outline

Green's dyadic

Green's dyadic: response of EM field to polarization

$$E_i(x) = \int d^4x' \Gamma_{ij}(x,x') P_j(x')$$

$$H_i(x) = \int d^4x' \Phi_{ij}(x,x') P_j(x')$$

Static situation: frequency decomposition

$$\Gamma_{ij}(x,x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \Gamma_{ij}(\vec{x},\vec{x}';\omega)$$

$$\Phi_{ij}(x,x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \Phi_{ij}(\vec{x},\vec{x}';\omega)$$

Maxwell's equations ($\varepsilon_0 = \mu_0 = 1$)

$$\mathrm{rot}\vec{E} = -\frac{\partial \vec{H}}{\partial t} \quad \Rightarrow \quad \varepsilon_{ijk}\partial_j \Gamma_{kl} = i\omega \Phi_{il}$$

$$\mathrm{rot}\vec{H} = \frac{\partial \left(\vec{E}+\vec{P}\right)}{\partial t} \quad \Rightarrow \quad -\varepsilon_{ijk}\partial_j \Phi_{kl} - i\omega\Gamma_{il} = i\omega\delta_{il}\delta(\vec{x}-\vec{x}')$$

$$\mathrm{div}\vec{H} = 0 \quad \Rightarrow \quad \partial_i \Phi_{ij} = 0$$

Solving for Green's dyadic

Redefining Γ:

$$\Gamma'_{il} = \Gamma_{il} + \delta_{il}\delta(\vec{x}-\vec{x}') \quad \Rightarrow \quad \partial_i \Gamma'_{ij} = 0$$

Taking the rotation of Maxwell's equations, we get

$$\left(\nabla^2+\omega^2\right)\Gamma'_{ij} = -(\partial_i\partial_j - \delta_{ij}\nabla^2)\delta(\vec{x}-\vec{x}')$$
$$\left(\nabla^2+\omega^2\right)\Phi_{ij} = i\omega\varepsilon_{ikj}\partial_k\delta(\vec{x}-\vec{x}')$$

This has to be solved with boundary conditions:
e.g. for a conducting boundary, tangential electric field vanishes on the surface

$$\varepsilon_{ijk} n_j \Gamma'_{kl}(\vec{x},\vec{x}';\omega)\big|_{\vec{x}\in\Sigma} = 0$$

Main advantage of method: explicit gauge invariance.

Computing the Casimir stress

The two-point functions of fields are

$$\left\langle E_i(x)E_j(x')\right\rangle = -i\Gamma_{ij}(x,x')$$
$$\left\langle H_i(x)H_j(x')\right\rangle = i\frac{1}{\omega^2}\varepsilon_{ikl}\partial_k\varepsilon_{jmn}\partial_k\Gamma_{mn}(x,x')$$
$$(\text{from } \varepsilon_{ikl}\partial_k E_l(x) = i\omega H_i(x)\ )$$

and the Maxwell stress tensor is

$$T_{ij} = E_iE_j - \frac{1}{2}\delta_{ij}\vec{E}^2 + H_iH_j - \frac{1}{2}\delta_{ij}\vec{H}^2$$

$\Rightarrow$ Casimir stress on the surface.

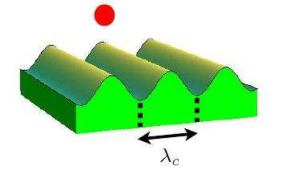
E.g. for a perfectly conducting sphere of radius a

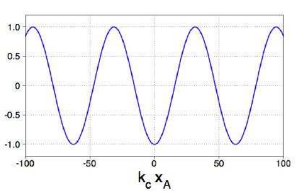
$$\mathscr{F} = \langle T_{rr}(r=a-0)\rangle - \langle T_{rr}(r=a+0)\rangle = \frac{1}{4\pi a^2}\left(-\frac{\partial E}{\partial a}\right)$$

and the self-energy from Casimir stress is (Boyer)

$$E = \frac{0.092353}{2a} \qquad (\hbar = 1 = c)$$

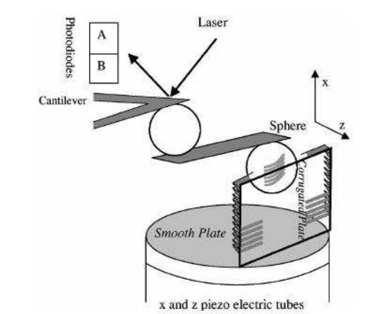
Outline

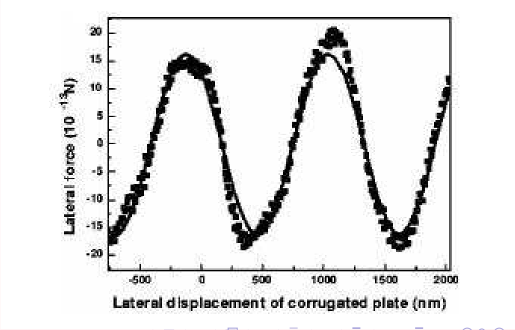




PFA: averaging over surface roughness. Condition: $\lambda_c \gg z_A$, zero lateral force.

F. Chen and U. Mohideen, Phys. Rev **A66**: 032113, 2002.



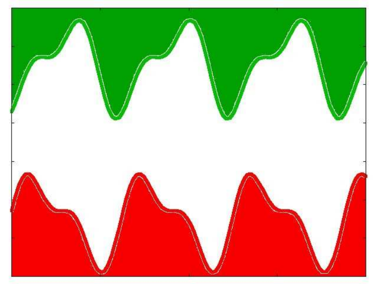


Use of Casimir force in micromachines

Standard worry: Casimir force would make nanobots stick.

Idea: exploit Casimir force to produce motion.

T. Emig: Casimir force driven ratchets
Phys. Rev. Lett. **98**:160801, 2007
[cond-mat/0701641]



A Casimir ratchet producing lateral motion by vibrating separation

With typical parameters $\langle v \rangle \sim$ mm/s

Other similar effect: Casimir torque (for asymmetric bodies)
Not yet observed!

Outline

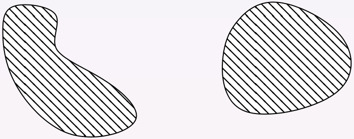
Arbitrary compact bodies

Emig, Graham, Jaffe & Kardar '2007

$$Z[\mathscr{C}] = \mathrm{Tr}\ e^{-\frac{i}{\hbar}H_{\mathscr{C}}T} = \int [\mathscr{D}\Phi]_{\mathscr{C}}\, e^{\frac{i}{\hbar}S[\Phi]}$$

$$\Phi(\vec{x}, t+T) = \Phi(\vec{x}, t)$$

$$\text{and } \Phi|_{\mathscr{C}} = 0$$



$$\mathrm{Tr}\ e^{-\frac{1}{\hbar}H_{\mathscr{C}}\Lambda} \underset{\Lambda\to\infty}{\to} e^{-\frac{1}{\hbar}\mathscr{E}_0[\mathscr{C}]\Lambda} + \ldots$$

$$\mathscr{C} = \bigcup_\alpha \Sigma_\alpha$$

$$\Rightarrow \mathscr{E}[\mathscr{C}] = \lim_{T\to -i\infty} \frac{\hbar}{|T|} \ln \frac{Z[\mathscr{C}]}{Z_\infty} = \sum_n \frac{\hbar}{2}(\omega_n - \omega_{n,\infty})$$

Suppose $\mathscr{C}$ is time-independent: we can Fourier expand in time

$$\int [\mathscr{D}\Phi]_{\mathscr{C}} \to \int \prod_n [\mathscr{D}\phi_n(\vec{x})]_{\mathscr{C}}$$

$$\Phi(x) = \sum_n \phi_n(\vec{x}) e^{2\pi i n t/T}$$

Fluctuating surface charges

So we get

$$\log Z[\mathscr{C}] = \sum_n \log\left\{ \int [\mathscr{D}\phi_n(\vec{x})]_{\mathscr{C}}\, e^{i\frac{T}{\hbar}\int d\vec{x}\left(\left(\frac{2\pi n}{cT}\right)^2|\phi_n(\vec{x})|^2 - |\nabla\phi_n(\vec{x})|^2\right)} \right\}$$

$$(T\to\infty) = \frac{cT}{\pi}\int_0^\infty dk \log \mathscr{Z}_{\mathscr{C}}(k)$$

$$\mathscr{Z}_{\mathscr{C}}(k) = \int [\mathscr{D}\phi(\vec{x},k)]_{\mathscr{C}}\, e^{\frac{i}{\hbar}T\int d^3\vec{x}\left(k^2|\phi(\vec{x},k)|^2 - |\nabla\phi(\vec{x},k)|^2\right)}$$

Now putting $T = -i\Lambda/c$, Wick rotating $k = i\kappa$

$$\mathscr{E}[\mathscr{C}] = -\frac{\hbar c}{\pi}\int_0^\infty d\kappa \log\frac{\mathscr{Z}_{\mathscr{C}}(i\kappa)}{\mathscr{Z}_{\infty}(i\kappa)}$$

$$\mathscr{Z}_{\mathscr{C}}(i\kappa) = \int [\mathscr{D}\phi(\vec{x},i\kappa)]_{\mathscr{C}}\, e^{-\frac{T}{\hbar}\int d^3\vec{x}\left(\kappa^2|\phi(\vec{x},i\kappa)|^2 + |\nabla\phi(\vec{x},i\kappa)|^2\right)}$$

Implement Dirichlet BC with Lagrange multipliers:

$$\int [\mathscr{D}\phi(\vec{x})]_{\mathscr{C}} = \int [\mathscr{D}\phi(\vec{x})] \underbrace{\prod_\alpha \int [\mathscr{D}\rho_\alpha(\vec{x})\mathscr{D}\rho^*_\alpha(\vec{x})]\, e^{i\frac{T}{\hbar}\int_{\Sigma_\alpha} d^3\vec{x}(\rho_\alpha(\vec{x})^*\phi(\vec{x}) + c.c.)}}_{\text{functional Dirac delta}}$$

Performing the Φ integral

So

$$\mathscr{Z}_{\mathscr{C}}(k) = \int [\mathscr{D}\phi(\vec{x},k)]_{\mathscr{C}} \prod_{\alpha} \int [\mathscr{D}\rho_\alpha(\vec{x})\mathscr{D}\rho^*_\alpha(\vec{x})]\, e^{\frac{i}{\hbar}T\tilde{S}(\phi,\rho)}$$

$$\tilde{S}(\phi,\rho) = \int d^3\vec{x}\left(k^2\,|\phi(\vec{x},k)|^2 - |\nabla\phi(\vec{x},k)|^2\right) + \int_{\Sigma_\alpha} d^3\vec{x}\,(\rho_\alpha(\vec{x})^*\phi(\vec{x},k) + c.c.)$$

Idea: integrate out Φ from quadratic functional integral $\rightarrow$ classical solution + fluctuations.

$$(\nabla^2 + k^2)\phi_{cl}(\vec{x},k) = 0 \; x \notin \Sigma_\alpha$$
$$\Delta\phi_{cl}(\vec{x},k) = 0 \; x \in \Sigma_\alpha$$
$$\Delta\partial_n\phi_{cl}(\vec{x},k) = \rho_\alpha(x) \; x \in \Sigma_\alpha$$

Integrating out fluctuations

$$\phi_{cl}(\vec{x}) = \sum_{\beta} \int_{\Sigma_\beta} d\vec{x}' \mathscr{G}_0(\vec{x}, \vec{x}', k) \rho_\beta(\vec{x}')$$

$$\mathscr{G}_0(\vec{x}, \vec{x}', k) = \frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} = ik \sum_{lm} j_l(kr_<) h_l^{(1)}(kr_>) Y_{lm}(\hat{x}') Y_{lm}(\hat{x})^*$$

Put now $\phi = \phi_{cl} + \delta\phi$

$$\mathscr{Z}_{\mathscr{C}}(k) = \prod_{\alpha} \int [\mathscr{D}\rho_\alpha(\vec{x}) \mathscr{D}\rho_\alpha^*(\vec{x})] \, e^{\frac{i}{\hbar} T \tilde{S}_{cl}(\rho)}$$

$$\times \underbrace{\int [\mathscr{D}\delta\phi(\vec{x}, k)] \, e^{i\frac{T}{\hbar}\int d^3\vec{x} \left(k^2 |\delta\phi(\vec{x},k)|^2 - |\nabla\delta\phi(\vec{x},k)|^2\right)}}_{\text{unconstrained fluctuations: cancel out with denominator}}$$

$$\tilde{S}_{cl}(\rho) = \int_{\Sigma_\alpha} d^3\vec{x} \left(\rho_\alpha(\vec{x})^* \phi(\vec{x}, k) + c.c.\right)$$

Also note that $\phi_{cl} = \sum_{\beta} \phi_\beta$, where ϕ_β is sourced by ρ_β.

Interaction terms

$$\phi_{cl}(\vec{x}) = \sum_{\beta} \int_{\Sigma_\beta} d\vec{x}' \left[ik \sum_{lm} j_l(kr_<) h_l^{(1)}(kr_>) Y_{lm}(\hat{x}') Y_{lm}(\hat{x})^* \right] \rho_\beta(\vec{x}')$$

Interaction terms ($\alpha \neq \beta$): in terms of multipoles

$$Q_{\beta,lm} = \int_{\Sigma_\beta} d\vec{x}_\beta j_l(kr_\beta) Y^*_{lm}(\hat{x}_\beta) \rho_\beta(\vec{x}_\beta)$$

$$\phi_\beta(\vec{x}_\beta) = ik \sum_{lm} Q_{\beta,lm} h_l^{(1)}(kr_\beta) Y_{lm}(\hat{x}_\beta)$$

$$\phi_\beta(\vec{x}_\alpha) = ik \sum_{lm} Q_{\beta,lm} \sum_{l'm'} \mathscr{U}^{\alpha\beta}_{lm,l'm'} h_{l'}^{(1)}(kr_\alpha) Y_{l'm'}(\hat{x}_\alpha)$$

$\mathscr{U}^{\alpha\beta}_{lm,l',m'}$: translation coefficients, depending on Σ_α and Σ_β

$$\begin{aligned} \tilde{S}_{\alpha\beta}(\rho) &= \int_{\Sigma_\alpha} d^3\vec{x} \left(\rho_\alpha(\vec{x})^* \phi_\beta(\vec{x}, k) + c.c. \right) \\ &= \frac{1}{2} ik \sum_{lm} \sum_{l'm'} \left(Q^*_{\alpha,l'm'} \mathscr{U}^{\alpha\beta}_{l'm',lm} Q_{\beta,lm} + c.c \right) \end{aligned}$$

Self-interaction terms

$$\tilde{S}_{\alpha\alpha}(\rho) = \frac{1}{2}\int_{\Sigma_\alpha} d^3\vec{x}\,(\rho_\alpha(\vec{x})^*\phi_\alpha(\vec{x},k) + c.c.)$$

Field inside Σ_α is regular Helmholtz solution, outside general

$$\phi_{in,\alpha}(\vec{x}) = \sum_{lm}\phi_{\alpha,lm}j_l(kr)Y_{lm}(\hat{x}) \qquad \phi_{out,\alpha}(\vec{x}) = \phi_{in,\alpha}(\vec{x}) + \Delta\phi_\alpha(\vec{x})$$

$$\Delta\phi_\alpha(\vec{x}) = \sum_{lm}\chi_{\alpha,lm}\left(j_l(kr)Y_{lm}(\hat{x}) + \sum_{l'm'}\mathscr{T}^{\alpha}_{l'm'lm}(k)h^{(1)}_{l'}(kr)Y_{l'm'}(kr)\right)$$

where $\mathscr{T}^{\alpha}_{l'm'lm}(k)$ is from $\Delta\phi_\alpha(\vec{x})|_{\Sigma_\alpha} = 0$. But the out field is regular at infinity $\Rightarrow \chi_{\alpha,lm} = -\phi_{\alpha,lm}$. So

$$\phi_{out,\alpha}(\vec{x}) = -\sum_{lm}\phi_{\alpha,lm}\sum_{l'm'}\mathscr{T}^{\alpha}_{l'm'lm}(k)h^{(1)}_{l'}(kr)Y_{l'm'}(kr)$$

but it is also $= \int_{\Sigma_\alpha} d\vec{x}'\mathscr{G}_0(\vec{x},\vec{x}')\rho_\alpha(\vec{x}') = ik\sum_{l'm'}Q_{\alpha,l'm'}h^{(1)}_{l'}(kr)Y_{l'm'}(\hat{x})$

so that $ikQ_{\alpha,l'm'} = \sum_{lm}\phi_{\alpha,lm}\mathscr{T}^{\alpha}_{l'm'lm}(k)$

$$\phi_{\alpha,lm} = -ik\sum_{l'm'}[\mathscr{T}^{\alpha}(k)]^{-1}_{l'm'lm}Q_{\alpha,l'm'}$$

Integrating over charge fluctuations

The final form for the self-interaction is

$$\tilde{S}_{\alpha\alpha}(\rho) = -\frac{ik}{2}\sum_{l'm'} Q_{\alpha,lm}\,[\mathscr{T}^{\alpha}(k)]^{-1}_{l'm'lm}\,Q_{\alpha,l'm'} + c.c.$$

and we are left with the functional integral

$$\begin{aligned}\mathscr{Z}_{\mathscr{C}}(k) &= \prod_{\alpha}\int[\mathscr{D}\rho_{\alpha}(\vec{x})\mathscr{D}\rho^{*}_{\alpha}(\vec{x})] \\ &\exp\left\{\frac{k}{2}\sum_{\alpha}\sum_{lm,l'm'} Q^{*}_{\alpha,lm}\left(\mathbb{T}^{-1}_{\alpha}\right)_{lm,l'm'} Q_{\alpha,l'm'}\right. \\ &\left. -\frac{k}{2}\sum_{\alpha\neq\beta}\sum_{lm,l'm'} Q^{*}_{\alpha,lm}\left(\mathbb{U}_{\alpha\beta}\right)_{lm,l'm'} Q_{\alpha,l'm'} - c.c.\right\} \\ &= \text{Jacobian} \times \prod_{\alpha,l.m}\left\{\int dQ_{\alpha,lm}\int dQ^{*}_{\alpha,lm}\right\}\exp\{\dots\}\end{aligned}$$

Jacobian is independent of functional integration variables ($Q-\rho$ relation linear) and drops out with denominator.

Casimir force: averaged interaction between fluctuating charges

The end result is:

$$E_{\mathscr{C}} = -\frac{\hbar c}{\pi}\int_0^\infty d\kappa \ln \frac{\det \mathbb{M}_{\mathscr{C}}(i\kappa)}{\det \mathbb{M}_\infty(i\kappa)}$$

$$\mathbb{M}(k) = \begin{pmatrix} \mathbb{T}_1^{-1} & \mathbb{U}_{12} & \cdots & \mathbb{U}_{1N} \\ \mathbb{U}_{21} & \mathbb{T}_2^{-1} & \cdots & \mathbb{U}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{U}_{N1} & \mathbb{U}_{N2} & \cdots & \mathbb{T}_N^{-1} \end{pmatrix} \quad \mathbb{M}_\infty(k) = \begin{pmatrix} \mathbb{T}_1^{-1} & 0 & \cdots & 0 \\ 0 & \mathbb{T}_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbb{T}_N^{-1} \end{pmatrix}$$

For two bodies:

$$E_{12}(\mathscr{C}) = -\frac{\hbar c}{\pi}\int_0^\infty d\kappa \operatorname{Tr} \ln \left(1 - \mathbb{T}^1\mathbb{U}^{12}\mathbb{T}^2\mathbb{U}^{21}\right)$$

Note: this is entirely finite, convergent and physically meaningful.

General formula for planar situations

In one space dimension it is easy to derive the Casimir interaction with other methods:

$$E_{12}(L) = -\frac{\hbar c}{\pi} \int_0^\infty d\kappa \log\left[1 - \mathrm{e}^{-2\kappa L} R_1(i\kappa) R_2(i\kappa)\right]$$

where $R_{1,2}(\omega)$ is the reflection coefficient of the mode ω on the boundaries and

$$\mathrm{e}^{-2\kappa L} = \mathrm{e}^{2i\omega L} = \mathrm{e}^{2i|k|L} \qquad , \qquad \omega = |k|$$

So here:

$$\mathbb{T}^1 = R_1(\omega) \qquad \mathbb{T}^2 = \mathrm{e}^{i\omega L} R_2(\omega)$$
$$\mathbb{U}^{12} = \mathbb{U}^{21} = \mathrm{e}^{2i\omega L}$$

which looks really sensible.
This also extends to planar situations

$$E_{12}(L) = -\frac{\hbar c}{\pi} \int_0^\infty d\kappa \int d\vec{k}_\perp \log\left[1 - \mathrm{e}^{-2L\sqrt{\kappa^2 + \vec{k}_\perp^2 + m^2}} R_1(i\kappa, \vec{k}_\perp) R_2(i\kappa, \vec{k}_\perp)\right]$$

(Bajnok, Palla & Takács, hep-th/0506089).

Outline

1. Introduction: QED and the Casimir effect
2. Realistic cases I: temperature and material dependence
3. Realistic cases II: geometry dependence
4. Comments on Casimir force and zero-point energy
   - Zero-point energy
   - Why does the ZPE derivation work?
   - Casimir force and van der Waals interaction
5. Time dependent boundaries
6. Gravitational aspects
7. Some related topics

Outline

Does the Casimir force originate from zero-point energy?

Mystery: a naive consideration of zero modes leads to a **huge** vacuum energy density.
Quantum field

$$\Phi(\vec{x},t) = \int \frac{d^d\vec{k}}{(2\pi)^{d/2}} \frac{1}{\sqrt{2\omega(\vec{k})}} \left(a(\vec{k}) e^{-i\omega(\vec{k})t + i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k}) e^{+i\omega(\vec{k})t - i\vec{k}\cdot\vec{x}} \right)$$

$$H = \int d^d\vec{x}\, T_{00} = \int d^d\vec{x} \frac{1}{2} (\partial_t \Phi)^2 + \frac{1}{2} \left(\vec{\nabla}\Phi\right)^2$$

$$= \int \frac{d^d\vec{k}}{(2\pi)^d} \omega(\vec{k}) \frac{1}{2} \left[a^\dagger(\vec{k}) a(\vec{k}) + a(\vec{k}) a^\dagger(\vec{k}) \right]$$

$$= \int \frac{d^d\vec{k}}{(2\pi)^d} \omega(\vec{k}) a^\dagger(\vec{k}) a(\vec{k}) + \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{2} \omega(\vec{k}) \delta(0)$$

With $\delta(0) = (2\pi)^d V$, $d = 3$ and a high energy cutoff Λ we get an energy density

$$\frac{E_0}{V} = \int_0^\Lambda k^2 dk \frac{1}{2} k \propto \Lambda^4$$

The naive vacuum energy density and the QFT Hamiltonian

QFT (Standard Model) valid at least up to $\Lambda \sim 1$ TeV: $\frac{E_0}{V} \sim 10^{47}\,\frac{\mathrm{J}}{\mathrm{m}^3}$
If $\Lambda = M_{\mathrm{Planck}} \sim 10^{19}$ GeV : $\frac{E_0}{V} \sim 10^{110}\,\frac{\mathrm{J}}{\mathrm{m}^3}$
How comes the Casimir force is such a small effect?
Crucial observation: quantum Hamiltonian is not uniquely fixed!
E.g.: why is the standard mass point Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{q})$$

Explanation: this comes from correspondence principle

$$\frac{d}{dt}\hat{\mathscr{O}} = \frac{i}{\hbar}[\hat{H},\hat{\mathscr{O}}] \qquad [\hat{q},\hat{p}] = i\hbar$$
$$\frac{d}{dt}\hat{q} = \frac{\hat{p}}{M} \qquad \frac{d}{dt}\hat{p} = -V'(\hat{q})$$

$\hbar \to 0$: $\hat{q}$,$\hat{p}$ commute $\Rightarrow$ simultaneously diagonalizable $\Rightarrow$ eigenvalues obey classical equations of motion.

The naive vacuum energy density and the QFT Hamiltonian

A perfectly good Hamiltonian for QFT is given by

$$H = \int d^d\vec{x}\, T_{00} = \int d^d\vec{x} : \frac{1}{2}(\partial_t \Phi)^2 + \frac{1}{2}\left(\vec{\nabla}\Phi\right)^2 :$$

$$= \int \frac{d^d\vec{k}}{(2\pi)^d}\omega(\vec{k})\frac{1}{2} : a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k}) := \int \frac{d^d\vec{k}}{(2\pi)^d}\omega(\vec{k})a^\dagger(\vec{k})a(\vec{k})$$

Moral: QFT does not predict vacuum energy density! Some other interaction is needed $\Rightarrow$ gravity.

Einstein's "greatest mistake":

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} \quad = \quad \frac{8\pi G}{c^4}T_{\mu\nu}$$

$$T^{(\lambda)\nu}_{\mu} = -\frac{c^4\lambda}{8\pi G}g^{\nu}_{\mu} = \mathscr{E} g^{\nu}_{\mu}$$

Cosmological constant: $p = -\mathscr{E}$. Present concordance cosmology (ΛCDM):

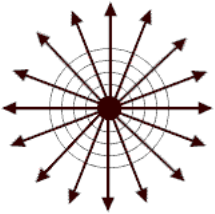
$$\mathscr{E} \sim 5.4\times 10^{-10}\,\frac{\mathrm{J}}{\mathrm{m}^3}$$

Outline

Why does the zero-point energy derivation work?

Energy of a point charge

$$E = \frac{e}{4\pi\varepsilon_0 r^2} \Rightarrow \mathscr{E} = \frac{1}{2}\varepsilon_0 \vec{E}^2 = \frac{e^2}{32\pi^2\varepsilon_0 r^4}$$



Field energy:

$$\int_{r_0}^{\infty} 4\pi r^2 \mathscr{E} \, dr = \frac{e^2}{8\pi\varepsilon_0 r_0}$$

$r_0 = 0$: divergent! Renormalization:

$$m_{\text{phys}} c^2 = m_0 c^2 + \frac{e^2}{8\pi\varepsilon_0 r_0}$$

m_{phys}: physical mass: the only observable.

Radius of the electron

Physical mass

$$m_{\rm phys}c^2 = m_0 c^2 + \frac{e^2}{8\pi\varepsilon_0 r_0}$$

$m_0 = 0$: classical electron radius

$$r_0 \sim 10^{-15} m$$

Present experiments: $r_0 < 10^{-18} m$

QED self-energy:

$$m_0 c^2 = m_{\rm phys}c^2 \left(1 - \frac{3\alpha}{4\pi}\log\left(\frac{\lambda^2_{\rm Compton}}{r_0^2} + \frac{1}{2}\right) + O(\alpha^2)\right)$$

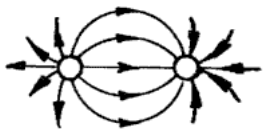
$$\lambda_{\rm Compton} = 2.4263102175(33) \times 10^{-12} m$$

$r_0 \sim 10^{-18} m$: 5% correction.

Theoretical limit: $m_0 > 0 \quad \rightarrow \quad r_0 > 10^{-136} m$

Two point charges

Figure : Two point charges with distance d



$$\vec{E} = \vec{E}_1 + \vec{E}_2 \quad \rightarrow \quad \mathcal{E} = \frac{1}{2}\varepsilon_0 \vec{E}^2$$

$$\mathrm{E}(d) = \int d^3\vec{x}\,\mathcal{E} \quad \text{still divergent for } r_0 = 0$$

$$\text{but: } \mathrm{E}(d_1) - \mathrm{E}(d_2) = \frac{e_1 e_2}{4\pi\varepsilon_0}\left(\frac{1}{d_1} - \frac{1}{d_2}\right) \qquad \text{finite!}$$

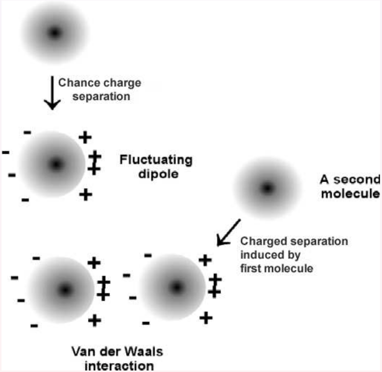
$$\text{Interaction energy:} \qquad \mathrm{E}_{int}(d) = \frac{e_1 e_2}{4\pi\varepsilon_0 d}$$

This works because

$$W_{\text{Lorentz}} = -\int d^3\vec{x}\,\Delta\mathcal{E}$$

Outline

van der Waals force = interaction between fluctuating dipols



$$H_{int} = \frac{\vec{d}_1 \cdot \vec{d}_2 r^2 - 3(\vec{d}_1 \cdot \vec{r})(\vec{d}_2 \cdot \vec{r})}{r^5}$$

$$V_{eff} = \sum_{m \neq 0} \frac{\langle 0|H_{int}|m\rangle\langle m|H_{int}|0\rangle}{E_0 - E_m} \propto r^{-6}$$

Original problem investigated by Casimir & Polder: retardation effects on vdW force

Dielectric ball: Casimir self-stress ≡ vdW forces

Casimir effect = relativistic vdW

Outline

Outline

Moving boundary

Plates: $K: \ x^3 = 0$ and $K': \ x^3 = vt$.
Solve for Dirichlet Green's function (scalar field):

$$\left(\partial_t^2 - \nabla^2\right) G(x,x') = -\delta(x-x')$$
$$G(x,x') = 0 \qquad x,x' \in K \text{ or } K'$$

Energy density

$$\langle 0|T_{00}(x)|0\rangle = \frac{1}{2}\sum_{k=0}^{3}\langle 0|\partial_k\Phi(x)\partial_k\Phi(x)|0\rangle = \frac{i}{2}\lim_{x'\to x}\sum_{k=0}^{3}\partial_k\partial_k' G(x,x')$$

Solution in $x^3 < 0$: using method of images

$$G^{>}(x,x') = \frac{i}{4\pi^2}\left[\frac{1}{(x-x')^2} - \frac{1}{(x-S_K x')^2}\right]$$
$$S_K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Moving boundary II

Solution for $x_3 > vt$: use Lorentz transform to get into system of K', find image, transform back.

$$G^{>}(x,x') = \frac{i}{4\pi^2}\left[\frac{1}{(x-x')^2} - \frac{1}{(x-S_{K'}x')^2}\right]$$

$$S_{K'} = \begin{pmatrix} \cosh s & 0 & 0 & -\sinh s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh s & 0 & 0 & -\cosh s \end{pmatrix} \qquad s = \log\frac{c-v}{c+c}$$

Solution in between: infinitely many images

$$G^{in}(x,x') = \frac{i}{4\pi^2}\sum_{m=-\infty}^{\infty}(-1)^m \frac{1}{(x-x'_m)^2}$$

$$x'_{2m} = (S_K S_{K'})^m x' \qquad x'_{2m-1} = S_K (S_K S_{K'})^m x'$$

$$x'_{-2m} = (S_{K'} S_K)^m x' \qquad x'_{-2m-1} = S_K (S_{K'} S_K)^m x'$$

Moving boundary III

Renormalization: eliminate vacuum contribution, which is the term

$$G_0 = \frac{i}{4\pi^2(x-x')^2}$$

in all three domains.

Force per unit area:

$$\mathscr{F}(a(t)) = -\frac{d}{d(vt)}\int_{-\infty}^{\infty} dx^3 \langle 0|T_{00}(x)|0\rangle \qquad a(t) = vt$$

$$= -\frac{\pi^2}{480a(t)^4}\left[1 + \frac{8}{3}\left(\frac{v}{c}\right)^2 + O\left(\frac{v^4}{c^4}\right)\right]$$

Electromagnetic case:

$$\mathscr{F}(a(t)) = -\frac{\pi^2}{240a(t)^4}\left[1 + \left(\frac{10}{\pi^2} - \frac{2}{3}\right)\left(\frac{v}{c}\right)^2 + O\left(\frac{v^4}{c^4}\right)\right] \qquad v \ll c$$

$$= -\frac{3}{8\pi^2 a(t)^4}\left[1 + \frac{(c^2-v^2)^2}{16c^4} + O\left(\frac{(c^2-v^2)^4}{c^8}\right)\right] \qquad v \ll c$$

Outline

Dynamical Casimir effect

Scalar field in $2d$

$$\partial_t^2 \Phi - c^2 \partial_x^2 \Phi = 0$$

Take an interval $(0, a(t))$, where $a(t) = a_0$ for $t < 0$. The field is

$$\Phi(t,x) = \sum_n \left(\chi_n^{(-)}(t,x) a_n + \chi_n^{(+)}(t,x) a_n^\dagger \right)$$

$$\chi_n^{(\pm)}(t \leq 0, x) = \frac{1}{\sqrt{\pi n}} e^{\pm i \omega_n t} \sin \frac{\pi n x}{a_0} \qquad \omega_n = \frac{c \pi n}{a_0}$$

$$\chi_n^{(-)}(t > 0, x) = \frac{1}{\sqrt{\pi n}} \sum_k Q_{nk}(t) \sqrt{\frac{a_0}{a(t)}} \sin \frac{\pi k x}{a(t)}$$

$$\chi_n^{(+)}(t > 0, x) = \left(\chi_n^{(-)}(t > 0, x) \right)^*$$

Initial conditions

$$Q_{nk}(0) = \delta_{nk} \qquad Q'_{nk}(0) = -i \omega_n \delta_{nk}$$

Equation of motion

Field equation gives

$$Q''_{nk}(t) + \omega_k^2(t) Q_{nk}(t)$$

$$= \sum_j h_{kj} \left[2\nu(t) Q'_{nj}(t) + \nu'(t) Q_{nj}(t) - \nu(t)^2 \sum_l h_{jl} Q_{nl}(t) \right]$$

$$\omega_k(t) = \frac{c\pi k}{a(t)} \qquad \nu(t) = \frac{a'(t)}{a(t)}$$

$$h_{kj} = -h_{jk} = (-1)^{k-j} \frac{2kj}{j^2 - k^2} \qquad j \neq k$$

Suppose that $a(T) = a_0$ after some time T $\Rightarrow$

$$t > T : Q_{nk}(t) = \alpha_{nk} e^{-i\omega_k t} + \beta_{nk} e^{i\omega_k t}$$

$$\Phi(t, x) = \sum_n \left(\phi_n^{(-)}(t, x) b_n + \phi_n^{(+)}(t, x) b_n^\dagger \right)$$

$$\phi_n^{(\pm)}(t, x) = \frac{1}{\sqrt{\pi n}} e^{\pm i\omega_n t} \sin \frac{\pi n x}{a_0} \qquad \omega_n = \frac{c\pi n}{a_0}$$

Bogolyubov transform

$$b_k = \sum_n \sqrt{\frac{k}{n}} \left(\alpha_{nk} a_n + \beta^*_{nk} a^\dagger_n\right)$$

$$\text{Unitarity: } \sum_k k\left(|\alpha_{nk}|^2 - |\beta_{nk}|^2\right) = n$$

In- and out-vacuum:

$$a_k|0\rangle_{in} = 0 \qquad b_k|0\rangle_{out} = 0$$

Number of created particles:

$$n_k = {}_{in}\langle 0|b^\dagger_k b_k|0\rangle_{in} = k\sum_{n=1}^{\infty} \frac{1}{n}|\beta_{nk}|^2$$

$$N = \sum_{k=1}^{\infty} n_k$$

Enhancing effect: parametric resonance. E.g.

$$a(t) = a_0\left[1 + \varepsilon \sin(2\omega_1 t)\right]$$

$$\omega_1 = \frac{c\pi}{a_0}$$

Particle creation

Solution is long, but result is that only odd modes are populated and

$$n_1(t) \approx \tau^2 \qquad \tau \ll 1$$

$$n_1(t) \approx \frac{4}{\pi^2}\tau \quad \tau \gg 1 \qquad \tau = \varepsilon\omega_1\tau$$

$$E(t) = \omega_1 \sum_k k n_k(t) = \frac{1}{4}\omega_1^2 \sinh^2(2\tau)$$

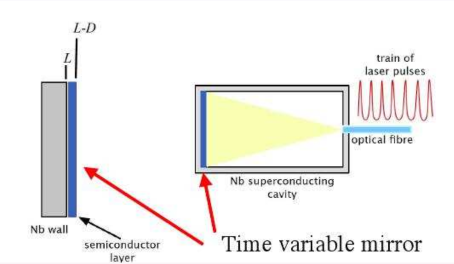
Typical values for photons in cm cavity $\omega_1 \sim 60\,\mathrm{GHz}$
maximum endurance for wall materials $\varepsilon_{max} \sim 3 \times 10^{-8}$

$$\frac{dn_1}{dt} \approx \frac{4}{\pi^2}\varepsilon_{max}\omega_1 \sim 700\ \mathrm{s}^{-1}$$

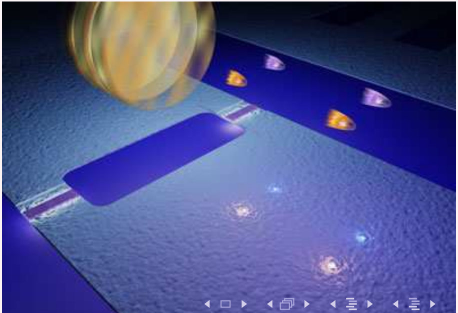
Total number created is typically thousands of photons per second.
Effects to take into account: finite wall reflectivity, detector interaction.
Nonzero temperature: factor $\sim 10^3$ at room temperature.

Experiments

MIR (Motion Induced Radiation, Padova) :(



Microwave line modulated by a SQUID: success!



C.M. Wilson et al., 2011
Nature **479**: 376-379
Microwave line: 100 μm
"Mirror motion": $\sim$ nm

Outline

Outline

Energy density

Scalar field, Dirichlet plates: Green's function of a given mode

$$g_{\text{int}}(z,z') = -\frac{1}{\lambda \sin\lambda a}\sin\lambda z_< \sin\lambda(z_> - a)$$

$$\Downarrow \qquad \lambda^2 = \omega^2 - k^2$$

$$\langle T_{00}\rangle = \frac{1}{2}\left[(\partial_t\Phi)^2 + (\nabla\Phi)^2\right] = \int \frac{d\omega d^2k}{(2\pi)^3}\langle t_{00}\rangle$$

$$\langle t_{00}\rangle = \frac{1}{2i}\left(\omega^2 + k^2 + \partial_z\partial_{z'}\right) g_{\text{int}}(z,z')|_{z=z'}$$

$$= -\frac{1}{2i\lambda\sin\lambda a}\left[\omega^2\cos\lambda a - k^2\cos\lambda(2z-a)\right]$$

Wick rotate $\omega \to i\zeta$, $\lambda \to i\kappa$ and use polar coordinates $\zeta = \kappa\cos\theta$, $k = \kappa\sin\theta$:

$$\langle T_{00}\rangle = -\frac{1}{4\pi^2}\int_0^\infty \kappa d\kappa \int_0^{\pi/2} d\theta\, \kappa^2 \frac{\sin\theta}{\sinh\kappa a}\Bigg[\cos^2\theta\cosh\kappa a$$

$$+\sin^2\theta\cosh\kappa(2z-a)\Bigg]$$

Energy density II

$$\langle T_{00} \rangle = -\frac{1}{6\pi^2} \int_0^\infty d\kappa \kappa^3 \left(\frac{1}{e^{2\kappa a} - 1} + \frac{1}{2} + \frac{e^{2\kappa z} + e^{2\kappa(a-z)}}{e^{2\kappa a} - 1} \right)$$

The second term is the vacuum constant, to be discarded. The result is

$$\langle T_{00} \rangle = u + g(z)$$

$$u = -\frac{\pi^2}{1440 a^4}$$

$$g(z) = -\frac{1}{6\pi^2} \frac{1}{16 a^4} \int_0^\infty dy y^3 \frac{e^{yz} + e^{y(1-z/a)}}{e^y - 1}$$

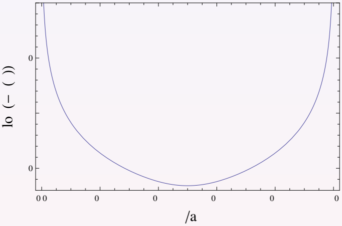
$$= -\frac{1}{16\pi^2 a^4} \left[\zeta(4, z/a) + \zeta(4, 1 - z/a) \right]$$

$$\zeta(s, z) = \sum_{n=0}^\infty \frac{1}{(n + a)^s} \qquad \text{Hurwitz zeta}$$

Energy density III

$g(z)$ diverges at $z = 0, a$. Fortunately

$$\int_0^a dz \left[e^{2\kappa z} + e^{2\kappa(a-z)} \right] == \frac{1}{\kappa} \left[e^{2\kappa a} - 1 \right]$$



so, although its integral is divergent, it is also a-independent and does not contribute to the force.

Similar calculation gives T_{xx}, T_{yy}, T_{zz}

$$\langle T^{\mu\nu} \rangle = u \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} + g(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Energy-momentum tensor

The energy-momentum tensor is not unique: instead of canonical we may use the conformal one

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{6}\left(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2\right)\Phi^2$$

for which

$$\tilde{T}^\mu_\mu = 0$$

Then

$$\langle T^{\mu\nu}\rangle = u\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad u = -\frac{\pi^2}{1440 a^4}$$

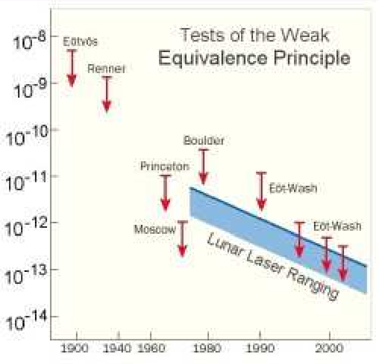
Casimir pressure and energy density

$$p = -3u$$

$$e = u$$

Outline

Equivalence principle



Binding energy: mass defect
Chemical bonds: $\Delta m/m = 10^{-9}$

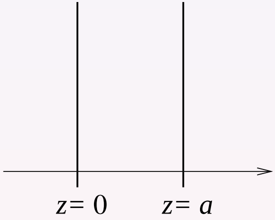
$\Downarrow$

The equivalence principle is valid for EM energy with at least 10^{-3} precision!

How does Casimir energy fall?

Between parallel plates

$$\langle T^{\mu\nu} \rangle = u \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 3 \end{pmatrix} \theta(z)\theta(a-z)$$



$$u = -\frac{\pi^2 \hbar c}{1440 a^4}$$

Remarks:

1. Volume divergence („ZPE") trivially eliminated.

$$u_0 = \frac{\hbar}{2} \int \frac{d^3\vec{k}}{(2\pi)^3} c \left| \vec{k} \right|$$

2. Surface divergence $\propto z^{-4} \Rightarrow$ renormalizing mass of plates.

Equivalence principle holds!

Gravitation energy in weak field limit:

$$E_g = -\int d^3\vec{x}\, h_{\mu\nu}(\vec{x})\, T^{\mu\nu}(\vec{x})$$

Problem: E_g is not gauge invariant!

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \;:\; \Delta E_g = 2\int d^3\vec{x}\, \xi_\mu \partial_\nu T^{\mu\nu}$$

Why? $\partial_\nu T^{\mu\nu} \neq 0$: there is a force on the plates!
Solution: Use locally inertial coordinates (K.A. Milton et al.):
Fermi coordinates: g_{ij} quadratic in distance from origin. Locally

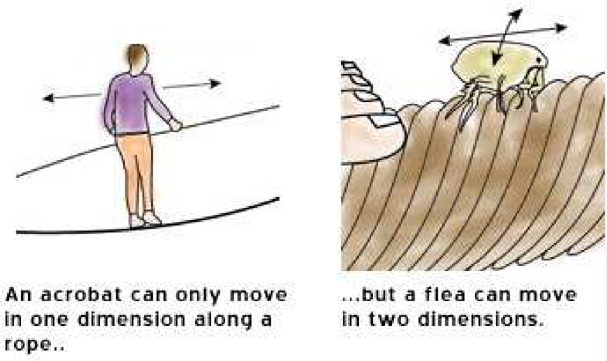
$$h_{00} = -gz \qquad h_{0i} = h_{ij} = 0$$
$$E_g = gz_0 uAa + \text{const} = gz_0 E_{\text{Casimir}} + \text{const}$$

which is just right!
A full analysis: K.A. Milton et al: *How does Casimir energy fall? IV*, arXiv:1401.0784

Outline

Compact extra dimensions

Compact extra dimensions: Kaluza-Klein theory, later resurrected by string theory.



$$\text{Space-time:} \qquad M = M_4 \times K$$

$$\langle T^{\mu\nu} \rangle = -u(a) g^{\mu\nu} = -\frac{\Lambda}{8\pi G} g^{\mu\nu}$$

Case of a sphere: $K = S^N$

Casimir energy of free massless scalar, for odd N

$$u(a) = -\frac{1}{64\pi^2 a^4}\mathrm{Re}\int_0^\infty dy[y^2 - i(N-1)y^2]D(iy)\frac{2\pi}{e^{2\pi y}-1}$$

$$D_l = \frac{(2l+N-1)(l+N-2)!}{(N-1)!l!}$$

$$N=1 : u(a) = -\frac{3\zeta(5)}{64\pi^6 a^4} \approx -\frac{5\times 10^{-5}}{a^4}$$

For even N $u(a)$ is logarithmically divergent; cutoff is necessary:

$$u(a) = \frac{1}{a^4}\left[\alpha_N \log\frac{a}{b} + \mathrm{const}\right]$$

$$\alpha_N = \frac{1}{16\pi^2}\mathrm{Im}\int_0^\infty \frac{dt}{e^{2\pi t}-1}[(N-1)it - t^2]^2 D(it)$$

b: frequency cut-off, presumably Planck scale. For large extra dimensions $a/b \sim 10^{16}$: logarithmic term sufficient for estimate.

Estimate for size of extra dimensions

Cosmological constant (ΛCDM concordance cosmology)

$$\Lambda \sim \rho_c \sim 10^{-5} \frac{\text{GeV}}{\text{cm}^3}$$

Maximum value for coefficient

$$u(a) \sim \frac{10^{-3}}{a^4}$$

Restoring units using $\hbar c = 2 \times 10^{-14}$GeV cm we find

$$a^4 \sim 10^2 \frac{\text{cm}^3}{\text{GeV}} \hbar c \sim 10^{-12} \text{cm}^4$$

$$a \sim 10\ \mu\text{m}$$

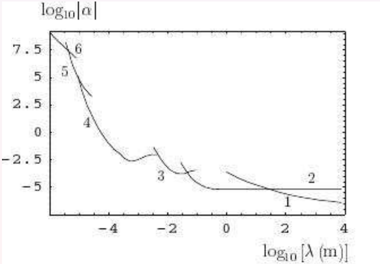
Such a compact dimension would lead to non-Newtonian gravity on a submm scale.

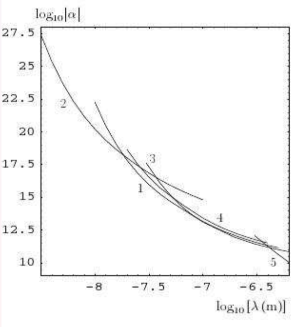
Outline

Non-Newtonian gravity experiments

E.g. searching for a correction of the form

$$V(r) = \alpha \frac{e^{-r/\lambda}}{r}$$





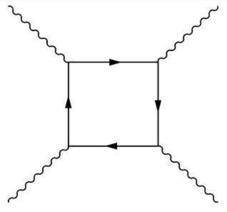
Presently: extra dimensions with size around 100 μm are ruled out.

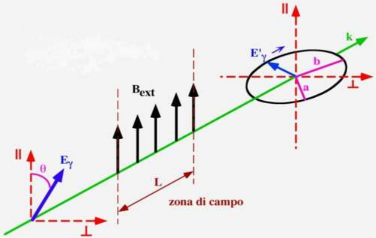
Outline

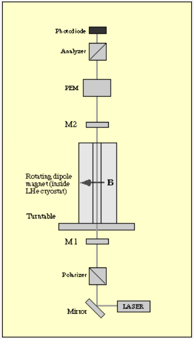
Outline

Vacuum birefringence

$$\mathscr{L}_{\text{effective}} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) + \frac{\xi}{2}\left(\left(\vec{E}^2 - \vec{B}^2\right)^2 + 7\left(\vec{E}\cdot\vec{B}\right)^2\right)$$







$$\xi = \frac{\hbar e^4}{45\pi m^4 c^7}$$

$$\Delta n \sim 4 \times 10^{-24} (B_{\text{ext}}/1\,\text{Tesla})^2$$

PVLAS (Polarizzazione del Vuoto con LASer, INFN, Padova)
G. Zavattini et al, QFEXT11, arXiv:1201.2309

Factor of 10^4 needed to reach sensitivity to QED: no signal yet!
$\rightarrow$ can still look for axion signal

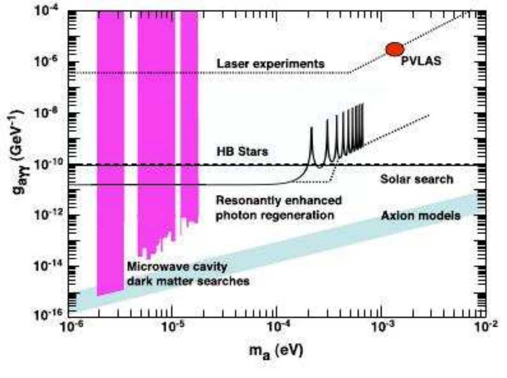
Outline

Axions

$$\mathscr{L} = \frac{1}{2}\partial_\mu a \partial^\mu a - \frac{1}{2} m_a^2 a^2 + \frac{1}{2}\left(\varepsilon \vec{E}^2 - \vec{B}^2\right) - g_a a \vec{E}\cdot\vec{B}$$

Axions induce vacuum birefringence

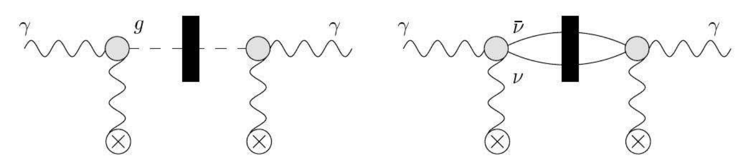
PVLAS had a signal, turned out to be detector effect on reanalysis



(2008 exclusion plot)

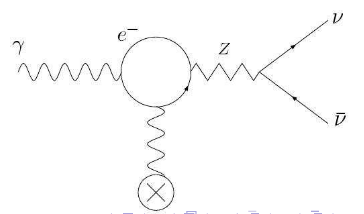
It is possible to shine light through walls using e.g. axions.

Standard modell contributions

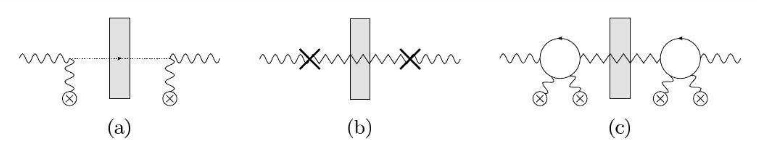


Graviton conversion very weak: $P(\gamma \to g \to \gamma) \sim 10^{-83} \left(\frac{B}{1\text{T}}\right)^4 \left(\frac{L}{1\text{m}}\right)^4$

Neutrino conversion is even weaker:



Shining light through walls: beyond the standard model



(a) Axions (b) Hidden sector γ (c) Hidden γ enhanced by MCP
(MCP: milli-charged particles)

ALP experiment (DESY), using HERA magnet

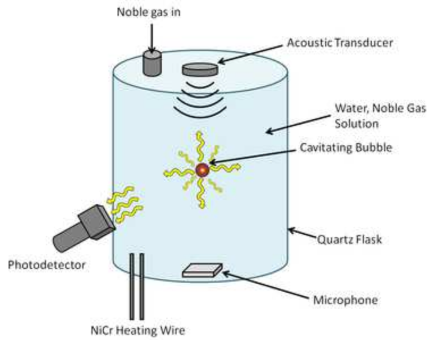
So far no signal...

J. Redondo and A. Ringwald: *Light shining through walls*, arXiv:1011.3741.

Outline

Sonoluminescence

Collapsing bubble emits flash of light
$a \sim 10^{-3}$ cm, overpressure ~ 1 atm,
$f \sim 10^4$ Hz, $E_{tot} \sim 10$ MeV



Schwinger: divergent bulk contribution

$$E_{bulk} = \frac{4\pi a^3}{3} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} k \left(1 - \frac{1}{n}\right)$$

Schwinger estimate (adiabatic approximation):

$$E_{bulk} \sim \frac{a^3 K^4}{12\pi} \left(1 - \frac{1}{\sqrt{\varepsilon}}\right)$$

Putting in $a \sim 4 \times 10^{-3}$ cm, cutoff $K \sim 2 \times 10^5$ cm^{-1} (UV),
$\sqrt{\varepsilon} \sim 4/3$:

$$E_c \sim 13 \text{ MeV}$$

Casimir calculations

Casimir energy for dielectric sphere (renormalized by bulk subtraction, equal to vdW!)

$$E = \frac{23}{1536\pi a}(\varepsilon - 1)^2 \qquad (|\varepsilon - 1| \ll 1)$$

Experiment: $a_i \sim 4 \times 10^{-3}$ cm to $a_f \sim 4 \times 10^{-4}$ cm

$$\Delta E \sim -10^{-4} \text{ eV}$$

Dynamical Casimir effect? Radiated energy spectrum: $T \sim 10^4$ K.
Simple estimate using results from Unruh effect:

$$\text{Unruh temperature: } T = \frac{\hbar A}{2\pi c} \qquad \text{Acceleration: } A \sim \frac{a}{\tau^2}$$

we get $\tau \sim 10^{-15}$ s which is way too short!
Experiment: collapse time scale 10^{-4} s, emission 10^{-11} s.
Best present explanation: towards end of bubble collapse $T \sim 10^4$ K, ionized noble gas radiates.

K.A. Milton, arXiv:hep-th/0009173