Gross-Pitaevskii theory

Outline

N non-interacting bosons

N interacting bosons, many-body Hamiltonian

Mean-field approximation, order parameter

Gross-Pitaevskii equation

Collapse for attractive interaction

Thomas-Fermi approximation for repulsive interaction

Collective excitations of the condensate

Small amplitude oscillations, Bogoliubov dispersion relation

Hydrodynamic equations, equation of continuity, Euler equation

Large amplitude oscillations, scaling equations

Vortices

N non-interacting bosons

Trapping potential

$$V_{\mathrm{ext}}(\mathbf{r}) = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$$

The many-body Hamiltonian is a sum of single-particle Hamiltonians whose eigenvalues have the form

$$\varepsilon_{n_x n_y n_z} = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z$$

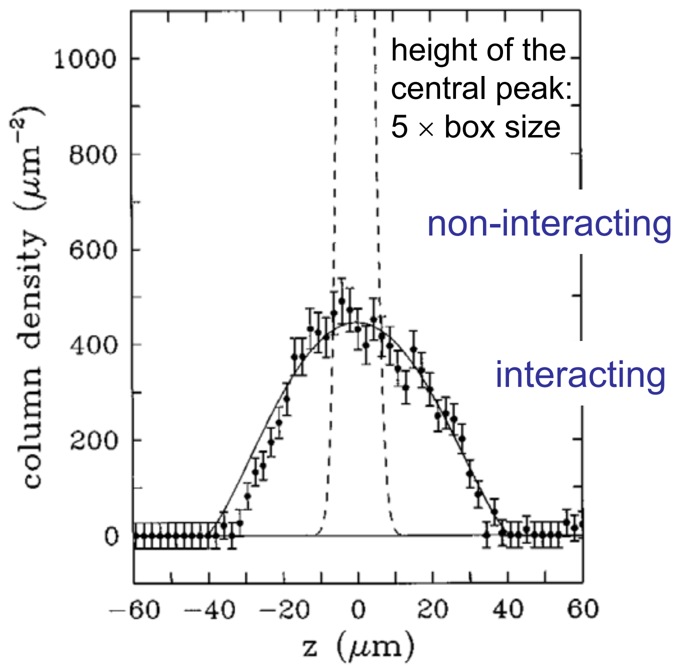
The ground state of N non-interacting bosons is given by putting all particles into the lowest single-particle state:

$$\phi(\mathbf{r}_1,\ldots,\mathbf{r}_N) = \Pi_i \varphi_0(\mathbf{r}_i)$$ with

$$\varphi_0(\mathbf{r}) = \left(\frac{m\omega_{\mathrm{ho}}}{\pi\hbar}\right)^{3/4} \exp\left[-\frac{m}{2\hbar}(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)\right]$$

$$\omega_{\mathrm{ho}} = (\omega_x \omega_y \omega_z)^{1/3}$$ (geometric average)

Density distribution of 80000 ^{23}Na atoms in a trap



Density distribution:

$$n(\mathbf{r}) = N|\varphi_0(\mathbf{r})|^2$$

Cloud size is independent of N and is fixed by the harmonic oscillator length:

$$a_{\mathrm{ho}} = \left(\frac{\hbar}{m\omega_{\mathrm{ho}}}\right)^{1/2}$$

N interacting bosons

Many-body Hamiltonian describing N interacting bosons confined by an external potential is given in second quantization by

$$\hat{H} = \int d\mathbf{r}\, \hat{\Psi}^\dagger(\mathbf{r}) \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\mathbf{r}) \right] \hat{\Psi}(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r}\, d\mathbf{r}'\, \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}') V(\mathbf{r}-\mathbf{r}') \hat{\Psi}(\mathbf{r}') \hat{\Psi}(\mathbf{r})$$

Field operator: $\hat{\Psi}(\mathbf{r}) = \Sigma_\alpha \Psi_\alpha(\mathbf{r}) a_\alpha$

single particle wave function

annihilation / creation operators:

$$a_\alpha | n_0, n_1, \ldots, n_\alpha, \ldots \rangle = \sqrt{n_\alpha}\, | n_0, n_1, \ldots, n_\alpha - 1, \ldots \rangle$$

$$a_\alpha^\dagger | n_0, n_1, \ldots, n_\alpha, \ldots \rangle = \sqrt{n_\alpha + 1}\, | n_0, n_1, \ldots, n_\alpha + 1, \ldots \rangle$$

The time evolution of the field operator of N interacting bosons is given in the Heisenberg representation by

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r},t) = [\hat{\Psi}, \hat{H}] = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{\mathrm{ext}}(\mathbf{r}) + \int d\mathbf{r}'\, \hat{\Psi}^\dagger(\mathbf{r}',t)\, V(\mathbf{r}'-\mathbf{r})\, \hat{\Psi}(\mathbf{r}',t) \right] \hat{\Psi}(\mathbf{r},t)$$

Solving the equation for more than 10^4 particles involves heavy numerical work.
Instead, we develop a mean-field theory that allows one also to understand the behavior of the interacting bose gas in terms of a set of parameters having a clear physical meaning.

Gross-Pitaevskii theory

The time evolution of the field operator of N interacting bosons is given in the Heisenberg representation by

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\mathbf{r},t)=[\hat{\Psi},\hat{H}]=\left[-\frac{\hbar^2\nabla^2}{2m}+V_{\mathrm{ext}}(\mathbf{r})+\int d\mathbf{r}'\hat{\Psi}^\dagger(\mathbf{r}',t)V(\mathbf{r}'-\mathbf{r})\hat{\Psi}(\mathbf{r}',t)\right]\hat{\Psi}(\mathbf{r},t)$$

Field operator: $\hat{\Psi}(\mathbf{r})=\Sigma_\alpha\Psi_\alpha(\mathbf{r})a_\alpha$

single particle wave function

annihilation / creation operators:

ground state wave function

$$\Psi_0=1/\sqrt{V}$$

$$a_\alpha|n_0,n_1,\ldots,n_\alpha,\ldots\rangle=\sqrt{n_\alpha}|n_0,n_1,\ldots,n_\alpha-1,\ldots\rangle$$

$$a_\alpha^\dagger|n_0,n_1,\ldots,n_\alpha,\ldots\rangle=\sqrt{n_\alpha+1}|n_0,n_1,\ldots,n_\alpha+1,\ldots\rangle$$

In a BEC macroscopic occupation of the ground state:

$\longrightarrow$ $N_0\pm1\simeq N_0$ thus $a_0=a_0^\dagger=\sqrt{N_0}$

Approximation of the field operator at very low T:

$\longrightarrow$ $\hat{\Psi}(\mathbf{r})=\sqrt{N_0/V}+\hat{\Psi}'(\mathbf{r})$

small perturbation (Bogoliubov 1947)

Gross-Pitaevskii equation

In general:

$$\hat{\Psi}(\mathbf{r},t)=\Phi(\mathbf{r},t)+\hat{\Psi}'(\mathbf{r},t) \quad \text{with} \quad \Phi(\mathbf{r},t)\equiv\langle\hat{\Psi}(\mathbf{r},t)\rangle \quad \text{and} \quad n_0(\mathbf{r},t)=|\Phi(\mathbf{r},t)|^2$$

classical field, order parameter, wavefunction of the condensate

This replacement is analogous to the transition from quantum electrodynamics to the classical description of electromagnetism in the case of a large number of photons.

Due to the large particle number the non-commutivity of the field operators is not important and the field can be described by classical functions.

Gross-Pitaevskii equation (1961)

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{r},t)=\left(-\frac{\hbar^2\nabla^2}{2m}+V_{\mathrm{ext}}(\mathbf{r})+g|\Phi(\mathbf{r},t)|^2\right)\Phi(\mathbf{r},t)$$

Replacement of the potential in the Heisenberg equation shall reproduce the same low energy scattering properties given by the bare potential V.

This is given by the pseudo-potential: $V(\mathbf{r}'-\mathbf{r})=g\,\delta(\mathbf{r}'-\mathbf{r})$ with $g=\dfrac{4\pi a\hbar^2}{M}$

Gross-Pitaevskii theory

The GP theory is a microscopic theory that describes the interacting non-uniform bose gas at zero temperature.

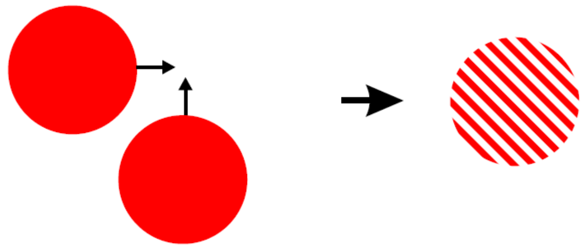
- GP describes BECs in traps that are non-uniform.
- GP exhibits new features in the dynamics as compared to the uniform case.

The GP theory is today the main tool for investigating trapped BECs. The GP equation has the form of a mean-field equation where the order parameter (condensate wavefunction) has to be calculated in a self-consistent way.

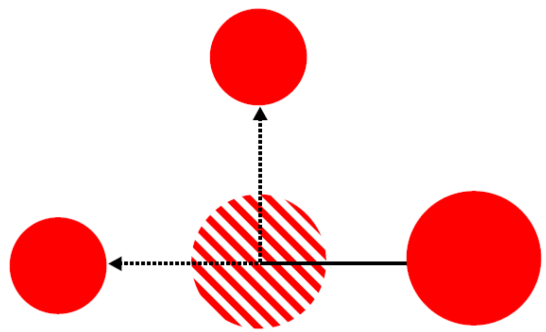
The GP theory describes typical properties of interacting condensates like propagation of excitations, interference, etc.

Four wave mixing

Another interpretation: Bragg scattering of a matter wave on a matterwave grating.



Density modulated grating at 45° represents a potential grating for the incoming wave due to the density dependent interaction potential.



The density modulation reflect the incoming wave with the same momentum by 90°.

Of course, the role of the two counter propagating initial waves can be interchanged.

Four wave mixing

Wave mixing with sum and difference frequency generation is known from nonlinear optics with intense laser fields.

The nonlinear interaction of the electric field with the mediums then described by:

$$\chi^{(3)}|E|^2 E$$

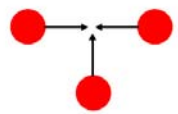
Similar nonlinear interaction is found in the GP-equation:

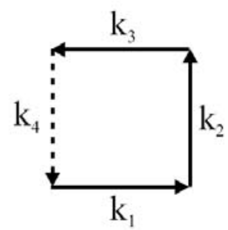
$$g|\Phi(\vec{r},t)|^2\Phi(\vec{r},t)$$

Difference: particle conservation in BEC!

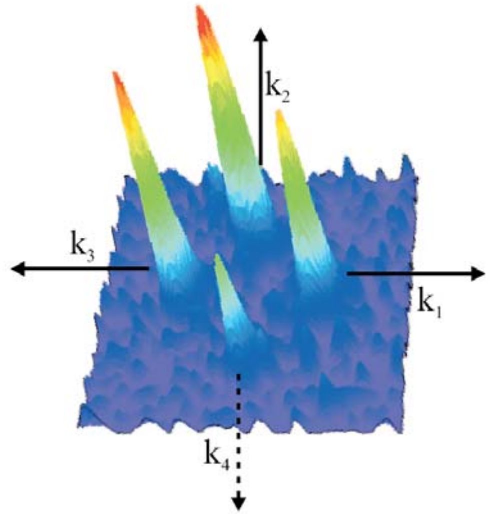
<u>Four wave mixing in BEC</u>:

Three incoming orthogonal matterwaves with momentum of equal amplitude and orthogonal direction generate a fourth wave.

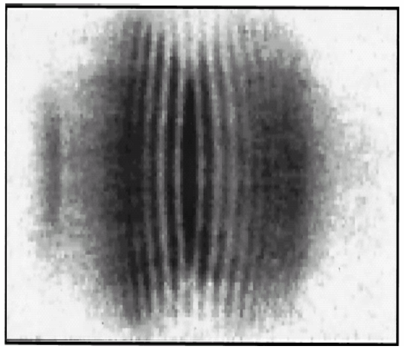




$$\vec{k}_4 = \vec{k}_1 - \vec{k}_2 + \vec{k}_3$$



Conservation laws associated with the GP equation



1. Particle number N due to

$$\begin{aligned}\int |\psi(\vec{r},t)|^2 \, d\vec{r} &= \int n(\vec{r},t) \cdot e^{i\varphi(\vec{r},t)} e^{-i\varphi(\vec{r},t)} d\vec{r} \\ &= \int n(\vec{r},t) d\vec{r} = N.\end{aligned}$$

The continuity equation is received by multiplying the GP equation by Ψ* and subtracting the complex conjugate of the resulting expression.

$$\frac{\partial n}{\partial t} + div \;\, j = 0$$ with $$\psi(\vec{r},t) = \sqrt{n(\vec{r},t)} \cdot e^{i\varphi(\vec{r},t)}$$

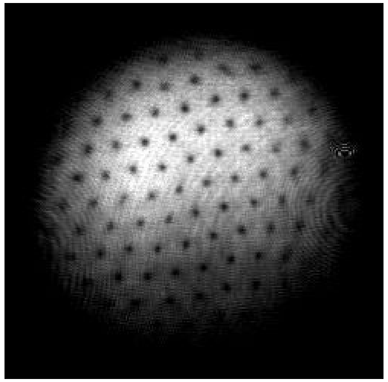
and the current density

$$\begin{aligned}n(\vec{r},t)v(\vec{r},t) &= \frac{\hbar}{2iM}\left(\psi^*(\vec{r},t)\vec{\nabla}\psi(\vec{r},t) - \psi(\vec{r},t)\vec{\nabla}\psi^*(\vec{r},t)\right) \\ &= \frac{\hbar}{2iM} \cdot 2in(\vec{r},t)\nabla\varphi(\vec{r},t) \\ &= \frac{\hbar}{M} n(\vec{r},t)\vec{\nabla}\varphi(\vec{r},t)\end{aligned}$$

The velocity field results to $$v(\vec{r},t) = \frac{\hbar}{M}\vec{\nabla}\varphi(\vec{r},t)$$

Conservation laws associated with the GP equation

The condensate is a liquid with a velocity potential



$$\vec{\nabla} \times v(\vec{r},t) = \frac{\hbar}{M} \vec{\nabla} \times \vec{\nabla} \varphi(\vec{r},t) = 0$$

$\rightarrow$ „irrotational flow“

(because decidedness of the phase, only quantized vortices are possible.)

2. The total energy is also conserved (for time independent potentials)

The time dependence of the order parameter is

$$\psi(r,t) = \psi(r)\, e^{-i\frac{\mu t}{\hbar}} \qquad \text{with} \qquad \mu = \frac{\partial E}{\partial N}$$

Separating the time evolution, we get the stationary GP equation

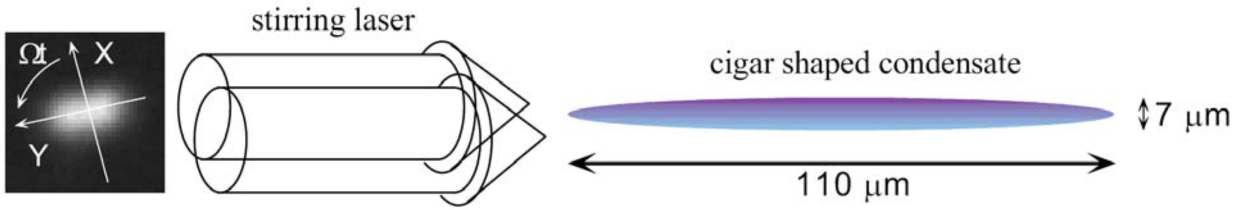
$$\mu\psi(\vec{r}) = \left(-\frac{\hbar^2 \nabla^2}{2M} + V(\vec{r}) + g\,|\psi(r,t)|^2 \right) \psi(\vec{r})$$

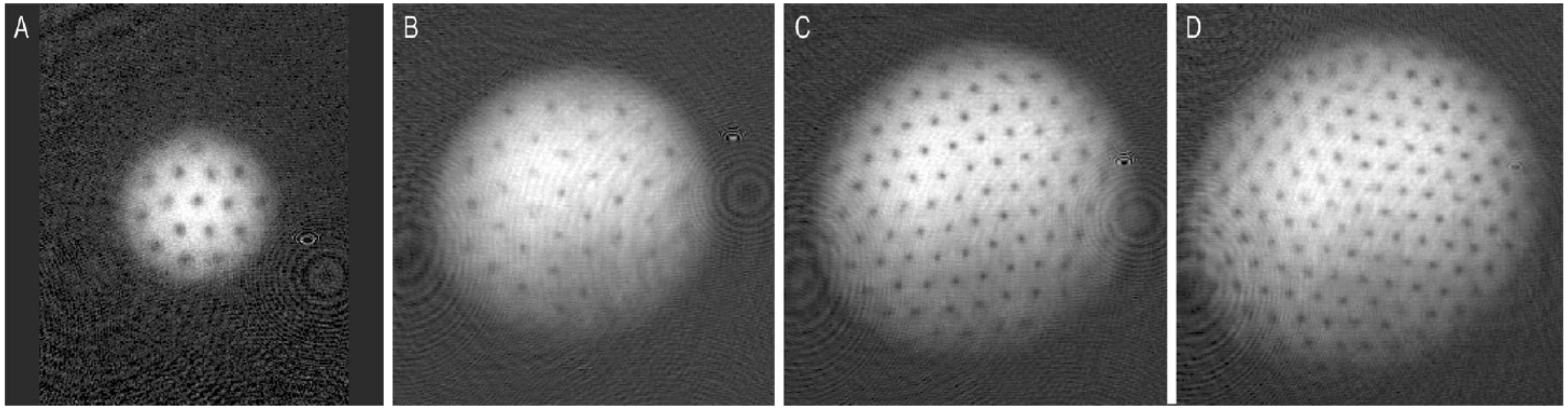
The value of chemical potential μ is given by the normalization condition:

$$E \;=\; \int \psi(\vec{r})^* \mu \psi(\vec{r}) d\vec{r} \;=\; \mu \int n(\vec{r}) d\vec{r} = N\mu$$

Creation of vortices in BEC

Stirring one/two laser beams:





J. Abo-Shaeer et al., Science 292, 476 (2001)

The examples show approximately (**A**) 16, (**B**) 32, (**C**) 80,and (**D**) 130 vortices. The vortices have „crystallized“ in a triangular pattern. The diameter of the cloud in (D) was 1 mm after ballistic expansion, which represents a magnification of 20.

Quantized vortices in the Gross-Pitaevskii theory

Quantized vortex along the *z* axis
is described by the order parameter: $\phi(\mathbf{r}) = \phi_v(r_\perp, z)\exp[i\kappa\varphi]$

κ is integer, φ *is* the angle around *z*, $\phi_v(r_\perp, z) = \sqrt{n(r_\perp, z)}$

Tangential velocity of the vortex state: $v = \dfrac{\hbar}{m r_\perp}\kappa$

Angular momentum along *z:* $N\kappa\hbar$

quantum of circulation

The GP equation takes the form:

$$\left[-\frac{\hbar^2\nabla^2}{2m} + \frac{\hbar^2\kappa^2}{2mr_\perp^2} + \frac{m}{2}(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2) + g\phi_v^2(r_\perp, z)\right]\phi_v(r_\perp, z) = \mu\phi_v(r_\perp, z)$$

centrifugal term

Non-interacting gas:

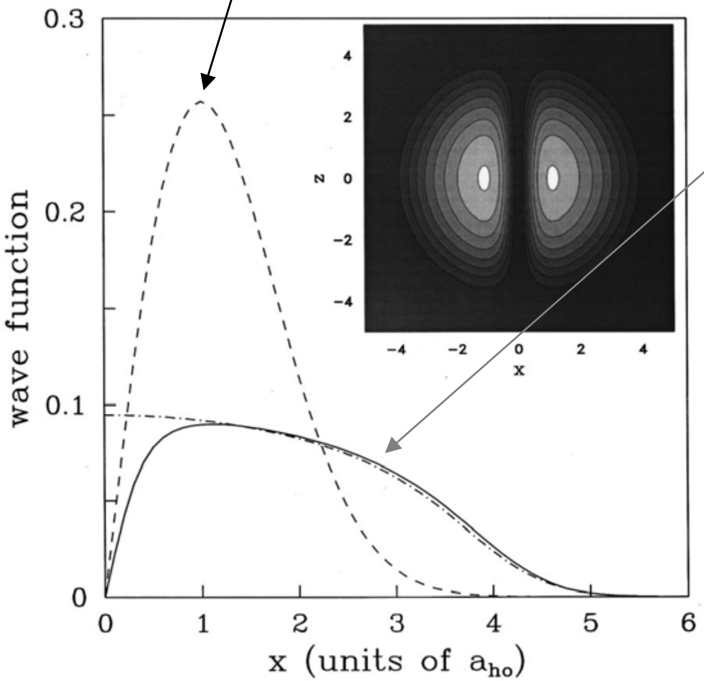
$$\phi_v(r_\perp, z) \propto r_\perp \exp\left[-\frac{m}{2\hbar}(\omega_\perp r_\perp^2 + \omega_z z^2)\right]$$

For $\kappa \neq 0$, Φ vanishes on the *z* axis.

Quantized vortices in the Gross-Pitaevskii theory

Noninteracting gas, for $\kappa = 1$: $\phi_v(r_\perp, z) \propto r_\perp \exp\left[-\frac{m}{2\hbar}(\omega_\perp r_\perp^2 + \omega_z z^2)\right]$

Energy of the state: $N\hbar\omega_\perp$ + ground state energy



BEC with repulsive interaction, $\kappa = 1$

Size of the core: $\xi = (8\pi n a)^{-1/2}$
(healing length)

Critical frequency for creating a vortex:

$$\Omega_C = \frac{E_0 - E_\kappa}{N\hbar\kappa} = \frac{5\hbar}{2mR_\perp^2} \ln \frac{0.671 R_\perp}{\xi}$$

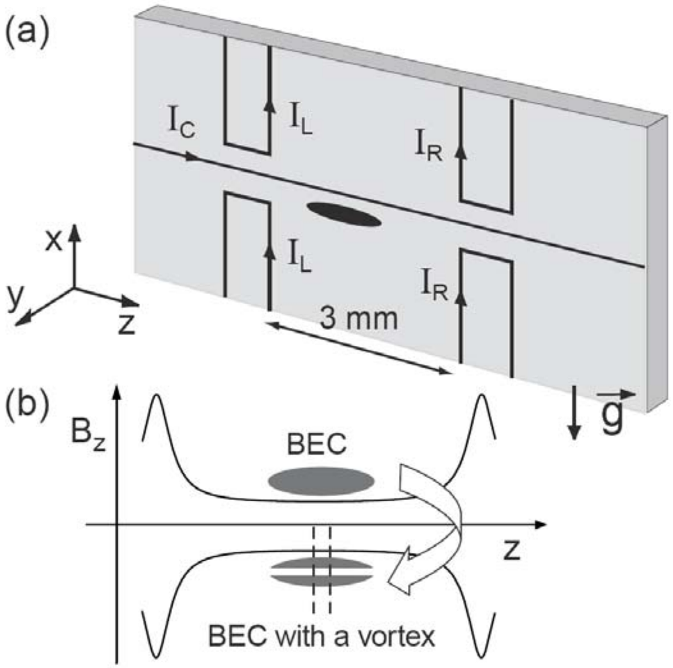
With $R_\perp$ the Thomas-Fermi radius of the condensate.

Creation of single quantized vortices is favored: $E_{kin} \propto (N\hbar\kappa)^2$

(kinetic E of atoms circulating around the core)

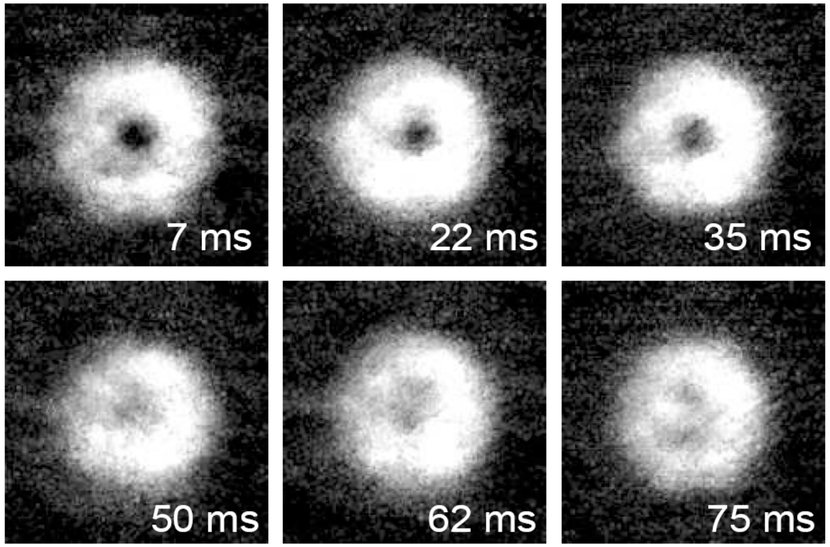
Creation of vortices in BEC

Topological vortices:



By inverting the z direction magnetic field, a doubly quantized vortex is imprinted.

Decay of double quantized vortices:



Axial absorption images of condensates after 15 ms of ballistic expansion with a variable hold time after imprinting a doubly quantized vortex.

A. E. Leanhardt et al., Phys. Rev. Lett. **89**, 190403 (2002)

Collapse for attractive forces (*a<0*)

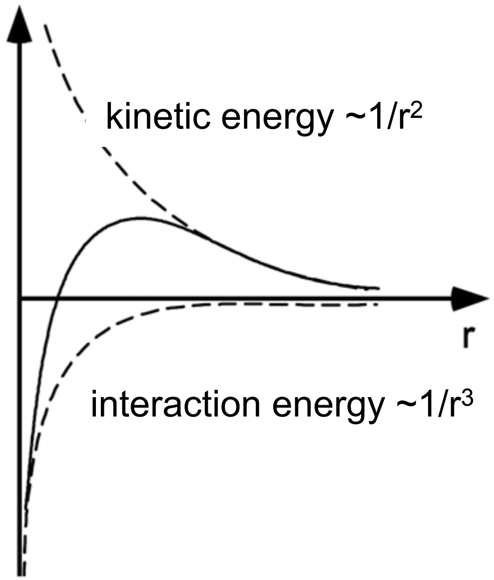
For attractive interaction the kinetic energy can not be neglected. It is stabilizing the condensate against collapses.

As long as interaction energy < kinetic energy the condensate is stable.

E_{kin} :
$$\left\langle -\frac{\hbar^2\nabla^2}{2M}\right\rangle \simeq -\frac{\hbar^2}{2M\Delta r^2}$$

E_{int} :
$$\left\langle g\left|\psi(r,t)\right|^2\right\rangle = gn(r) \simeq g\frac{N}{V} = g\frac{N}{\frac{4}{3}\pi\Delta r^3}$$

E_{kin} +E_{int} :
$$\langle\mu\rangle \simeq -\frac{\hbar^2}{2M}\frac{2\pi}{\Delta r^2} + N\frac{g}{\frac{4}{3}\pi\Delta r^3}.$$



Critical atom number in a harmonic trap

$$N_c = \frac{8\pi^2}{9}\frac{\hbar^{5/2}}{\omega^{1/2}\left|g\right| M^{3/2}} = \frac{2\pi}{9}\frac{1}{a}\sqrt{\frac{\hbar}{\omega M}}$$ with $$\Delta r = \sqrt{\frac{\hbar}{M\omega}}$$

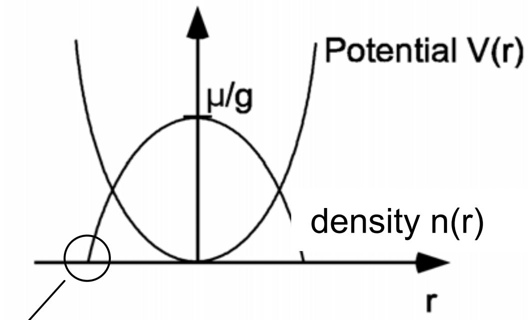
For $N>N_C$ collaps!

Thomas-Fermi approximation (*a*>0)

We approximate the GP equation for repulsive interaction (*a*>0) in the limit

interaction energy >> kinetic energy

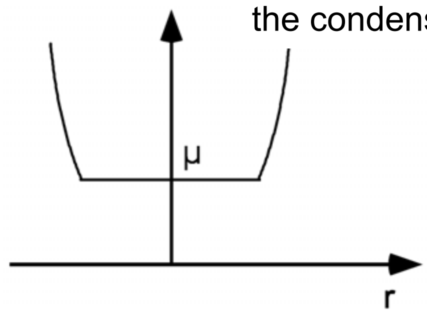
$$\begin{aligned}\mu\psi(\vec{r}) &= \left(-\frac{\hbar^2\nabla^2}{2m} + V(\vec{r}) + g\,|\psi(r,t)|^2\right)\psi(\vec{r}) \\ &\simeq \left(V(\vec{r}) + g\,|\psi(r,t)|^2\right)\psi(\vec{r})\end{aligned}$$



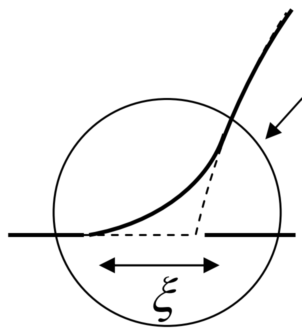
Thereby takes the density the shape of the potential:

$$n(r) = |\psi(r,t)|^2 = \frac{1}{g}\left(\mu - V(\vec{r})\right)$$

single particle potential in the condensate



$$ng \quad \leftrightarrow \quad \xi = \frac{1}{\sqrt{8\pi n a}}$$

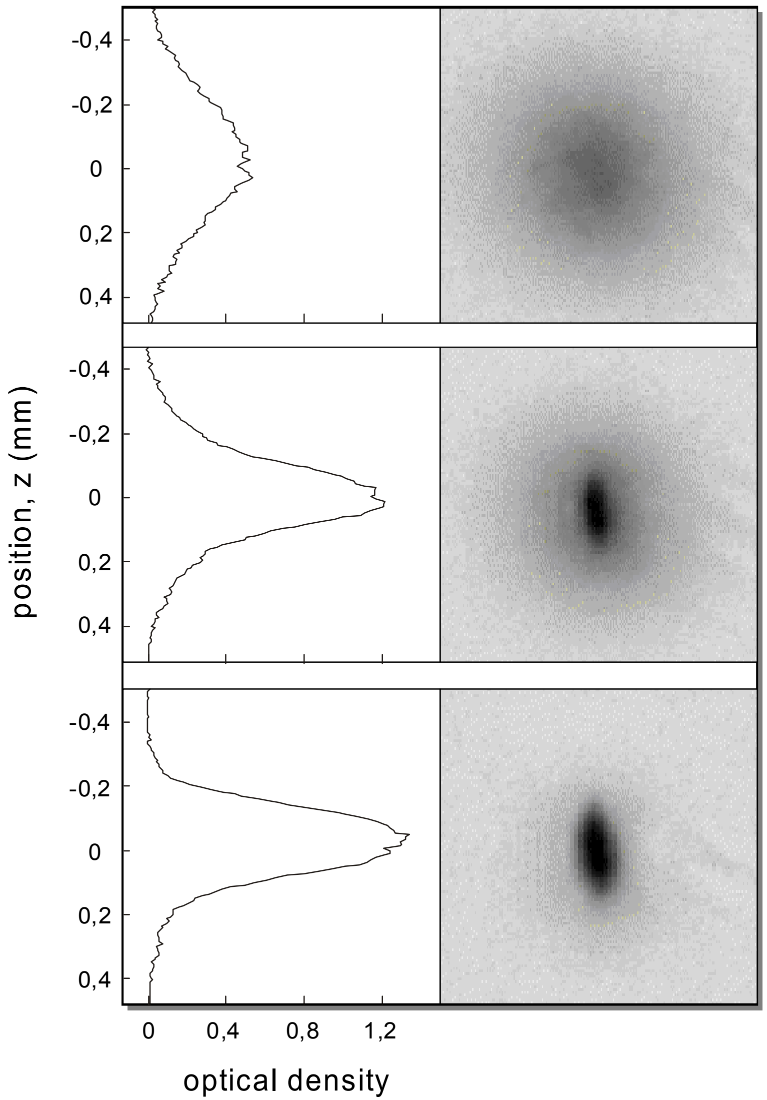


int. energy healing length

Bose-Einstein condensation

absorption images
after

20 ms ballistic expansion



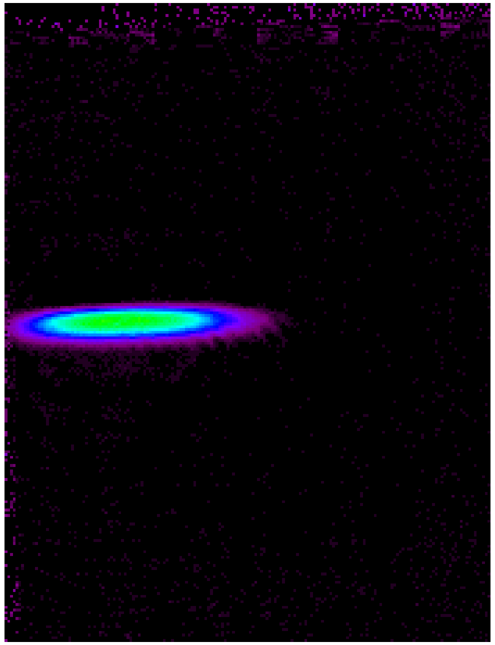
T= 1 μK

T= 800 nK

T= 500 nK

The expansion is described by the scaling equations.
Castin and Dum
PRL 77, 5316 (1996)

Ballistic expansion of a condensate



scaling parameters

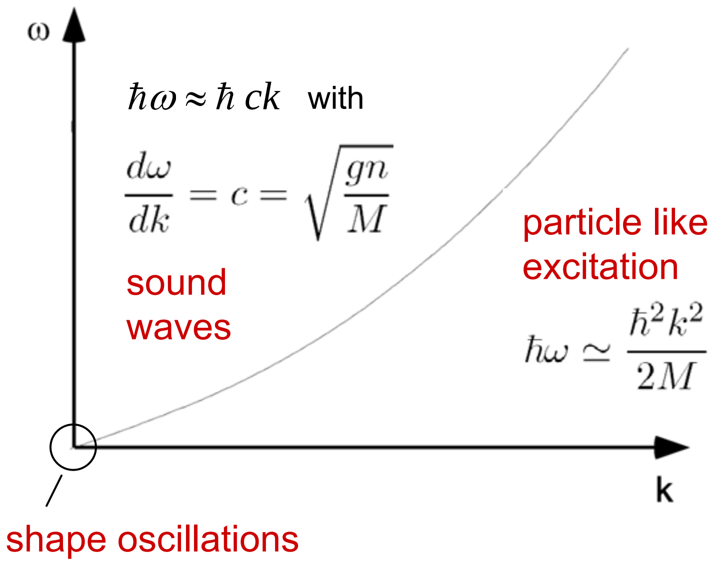
describing the radial and axial size of the condensate at any time in units of the Thomas-Fermi radius

$$\begin{aligned} b_\perp &= \sqrt{1+\tau^2} \\ b_{||} &= 1+\lambda^2\left(\tau\arctan(\tau) - \ln\sqrt{1+\tau^2}\right) \end{aligned} \qquad \tau := t\cdot\omega_\perp$$

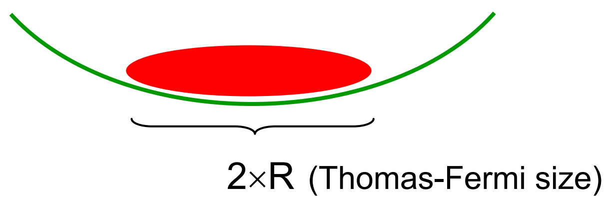
The phase of the condensate takes a parabolic profile and its evolution is given also by the scaling parameter (see Castin and Dum PRL 77, 5316 (1996)).

Collective excitations of a condensate (T=0)

Oscillations having wavelengths much smaller than the size of the condensate ($\omega \gg \omega_{trap}$) propagate as usual sound waves or free particles according to the Bogoliubov dispersion relation:



$$(\hbar\omega)^2 = \frac{\hbar^2 k^2}{2M}\left(\frac{\hbar^2 k^2}{2M} + 2gn\right)$$



Conversely, excitations at lower frequencies ($\omega \sim \omega_{trap}$) involve a motion of the whole condensate: Center of mass oscillations or collective shape oscillations. These can be derived from the hydrodynamic theory of superfluids in the collisionless regime.

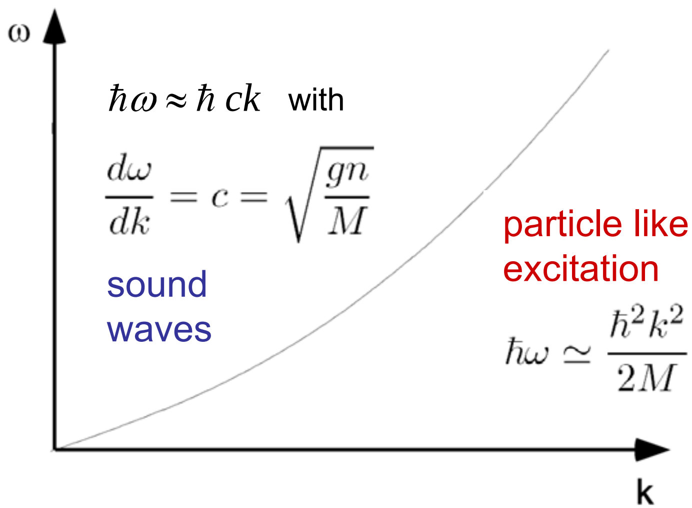
Bogoliubov dispersion relations

It is possible to excite sound waves and shape oscillations in the condensate at T=0. The dispersion relation $\omega(k)$ can be derived from the GP equation by the following ansatz: (Pitaevskii 1961)

$$\psi(r,t) = e^{-i\mu t/\hbar}\left(\psi(r) + u(r)e^{-i\omega t} + q^*(r)e^{i\omega t}\right)$$

u(r) and *q*(r)* describe small amplitude, harmonic perturbations on the condensate ground state.

After substitution into the GP equation, we get two coupled differential equations for *u(r)* and *q(r).* Taking just the linear terms into account, and with $\mu = ng$ the dispersion relation reads:

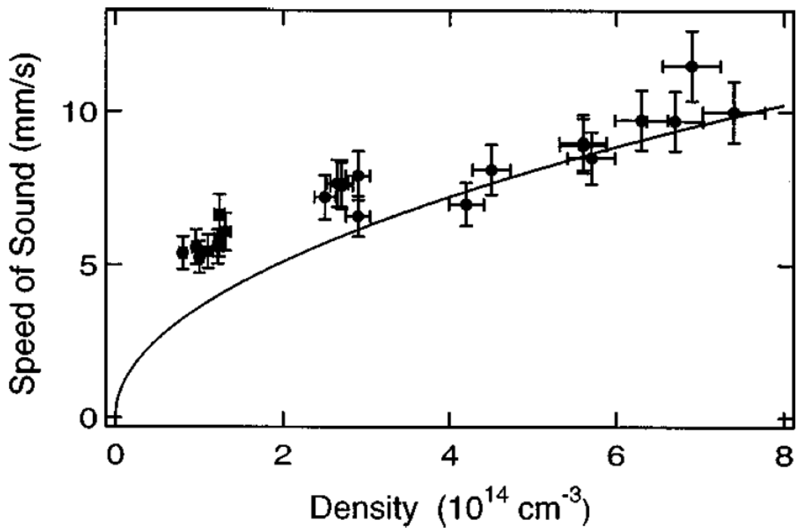


$$(\hbar\omega)^2 = \frac{\hbar^2 k^2}{2M}\left(\frac{\hbar^2 k^2}{2M} + 2gn\right)$$

(Bogoliubov 1947; second quantization, and diagonalization of the Hamiltonian)

Speed of sound

A sound wave is generated by focusing a laser pulse into the center of the condensate. A wave excitation formed in this way propagates and its position inside the condensate is detected at different times.



$$c = \sqrt{\frac{gn}{M}}$$

FIG. 15. Speed of sound c versus condensate peak density $n(0)$ for waves propagating along the axial direction in the cigar-shaped condensate at MIT. The experimental points are compared with the theoretical prediction $c=[gn(0)/2m]^{1/2}$ (solid line). From Andrews, Kurn *et al.* (1997).

Bogoliubov dispersion relation

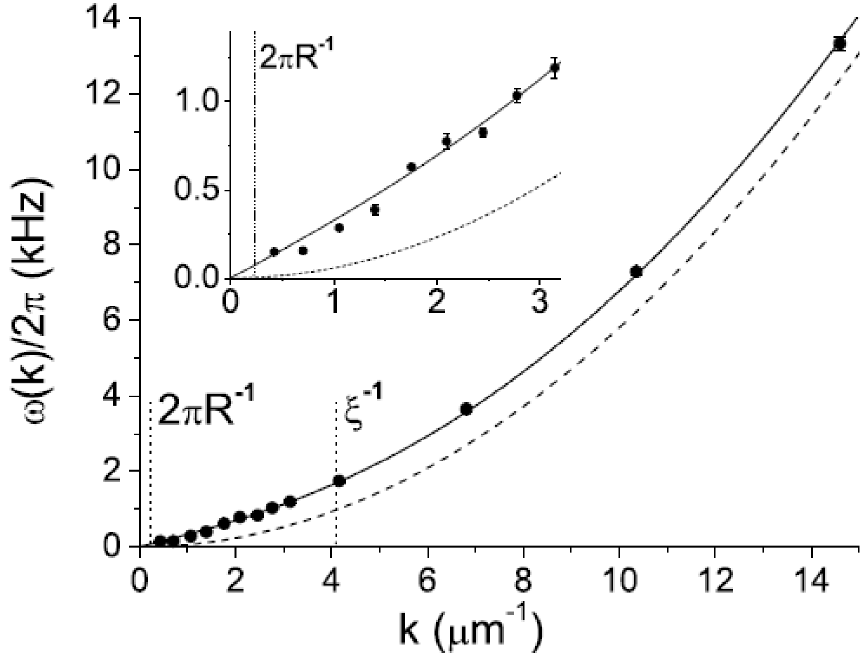
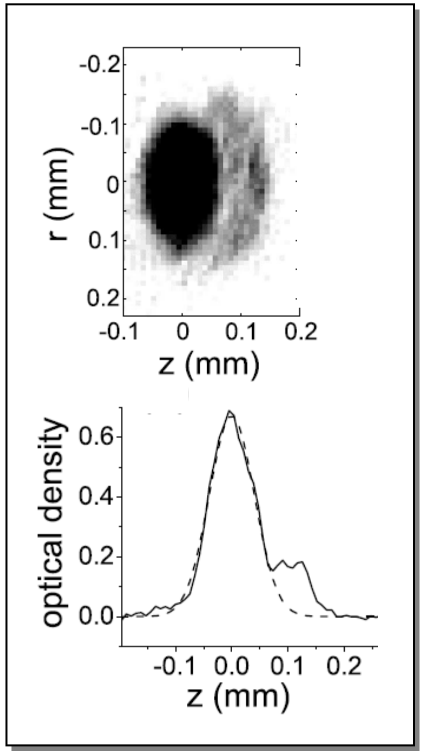


FIG. 3. (a) The measured excitation spectrum $\omega(k)$ of a trapped Bose-Einstein condensate. The solid line is the Bogoliubov spectrum with no free parameters, in the LDA for $\mu = 1.91$ kHz. The dashed line is the parabolic free-particle spectrum. For most points, the error bars are not visible on the scale of the figure. The inset shows the linear phonon regime.

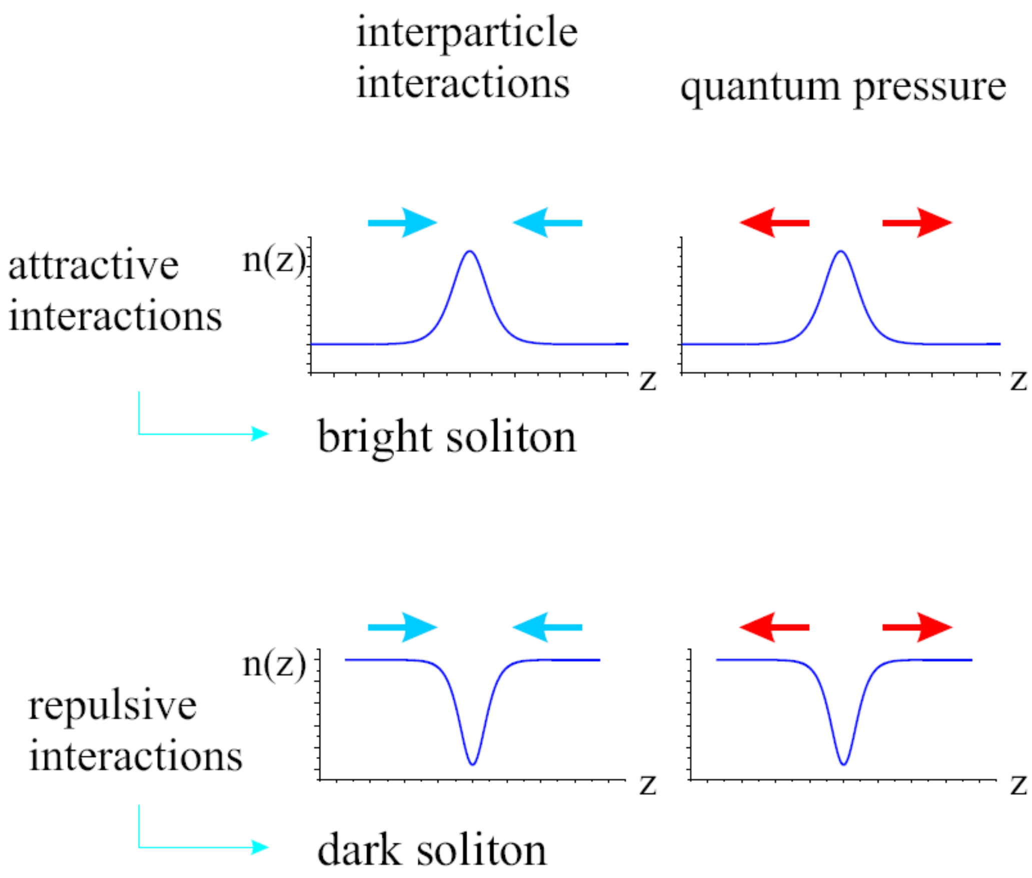
Measurements using „Bragg spectroscopy"



Excitation Spectrum of a BEC
J. Steinhauer, R. Ozeri, N. Katz, and N. Davidson
Phys. Rev. Lett. 88, 120407 (2002)

Solitons

Wavepacket like excitations of one dimensional nonlinear systems.



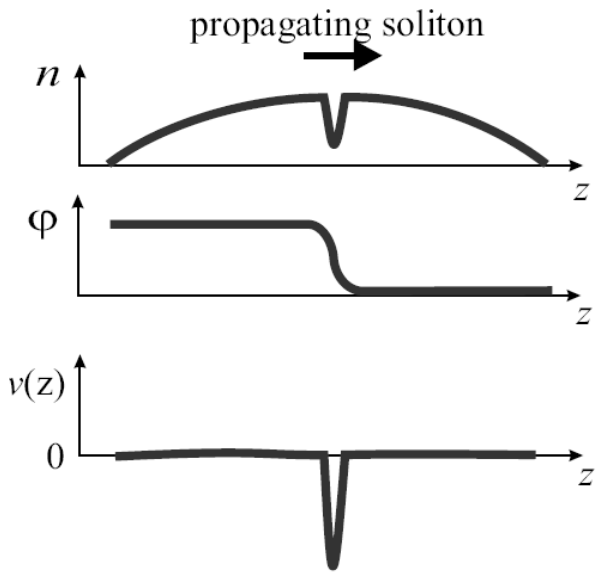
M. Weidemüller, C. Zimmermann (Eds.) „Interaction in ultracold gases“, Whiley-VCH 2003, Weinheim

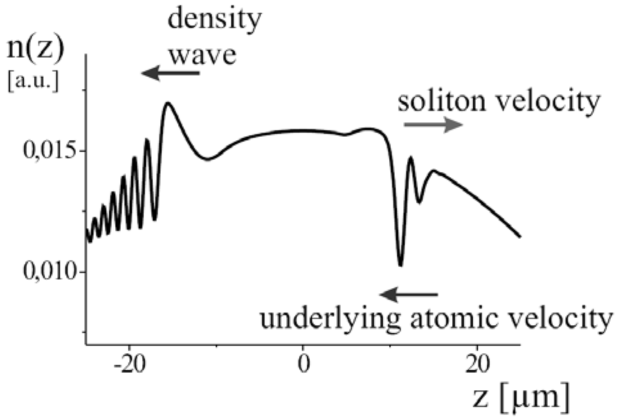
Dark solitons in BEC with *a>0*

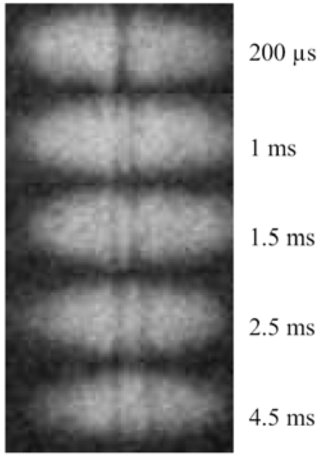
Phase imprinting, e.g. with applying an optical potential to one part of the condensate.

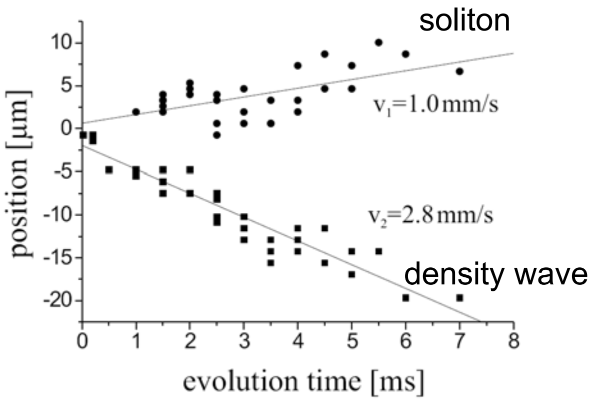
$$\varphi = \frac{U}{\hbar} t_{imprint}$$

For a pure phase imprinting (no density redistribution), $t_{imprint}$ has to be shorter than the correlation time $t_c=\mu/\hbar$ of the condensate.









The soliton propagates with a velocity smaller than the sound velocity (density wave).

Role of dimensionality

The dynamics of the condensate significantly changes with the dimensionality.

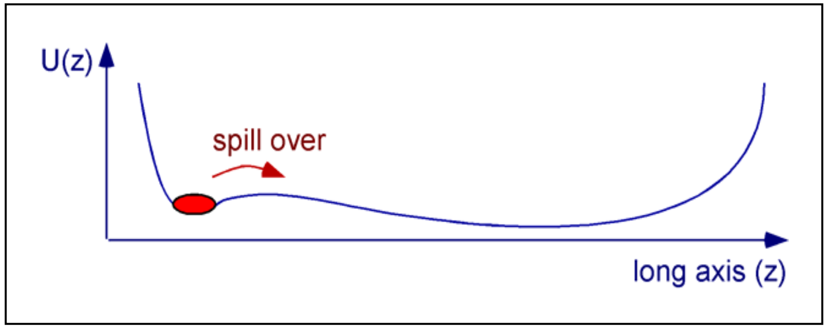
Cross-over from 3D to 1D:

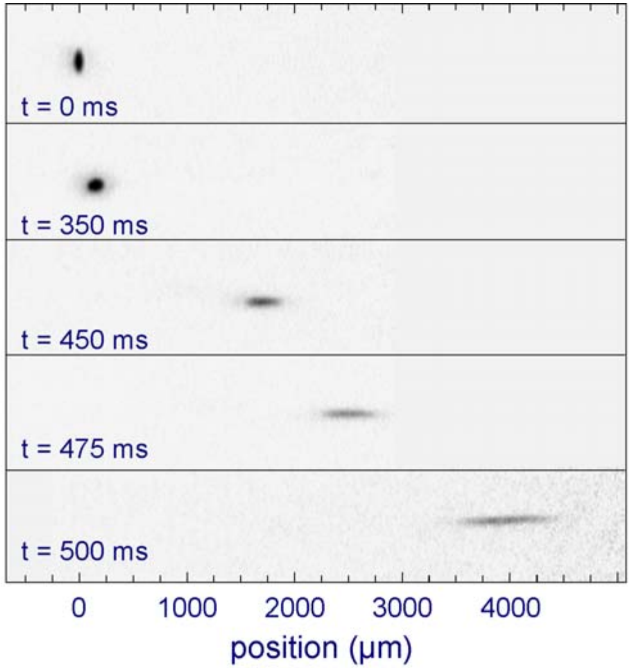
$k_B T > n\,g > \hbar\,\omega_r$	3D, quasi homogeneous, Thomas-Fermi regime
$k_B T > \hbar\,\omega_r > n\,g$	confined regime, BEC confined to the radial trap ground state, thermal cloud 3D
$\hbar\,\omega_r > k_B T > n\,g$	quasi 1D regime, two dimensional freeze out

…

The dynamics in 1D, 2D, and cross-over regimes are topic of a large number of theoretical and experimental work.

Expansion of a condensate in a waveguide





axial relaxation reduces interaction energy

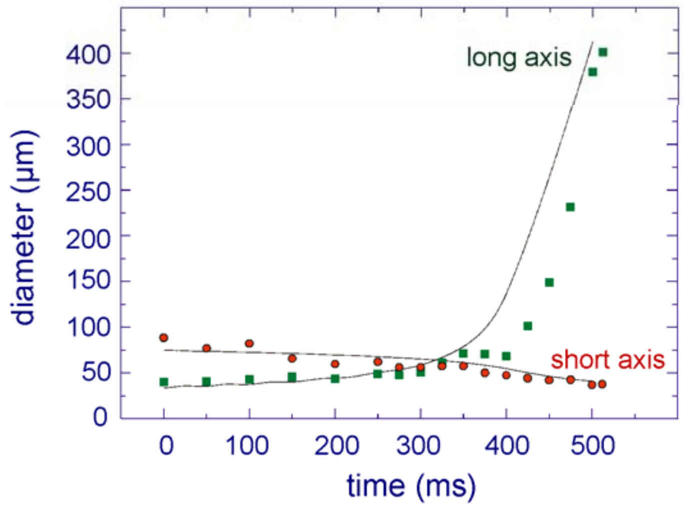
↓

"quasi-1D condensate" without excitations

$$i\hbar \frac{\partial}{\partial t}\phi(r,t) = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(r) + g|\phi(r,t)|^2\right)\phi(r,t)$$

wave packet propagating in transverse ground state

v=50 mm/s, Δv=5 mm/s



Role of dimensionality

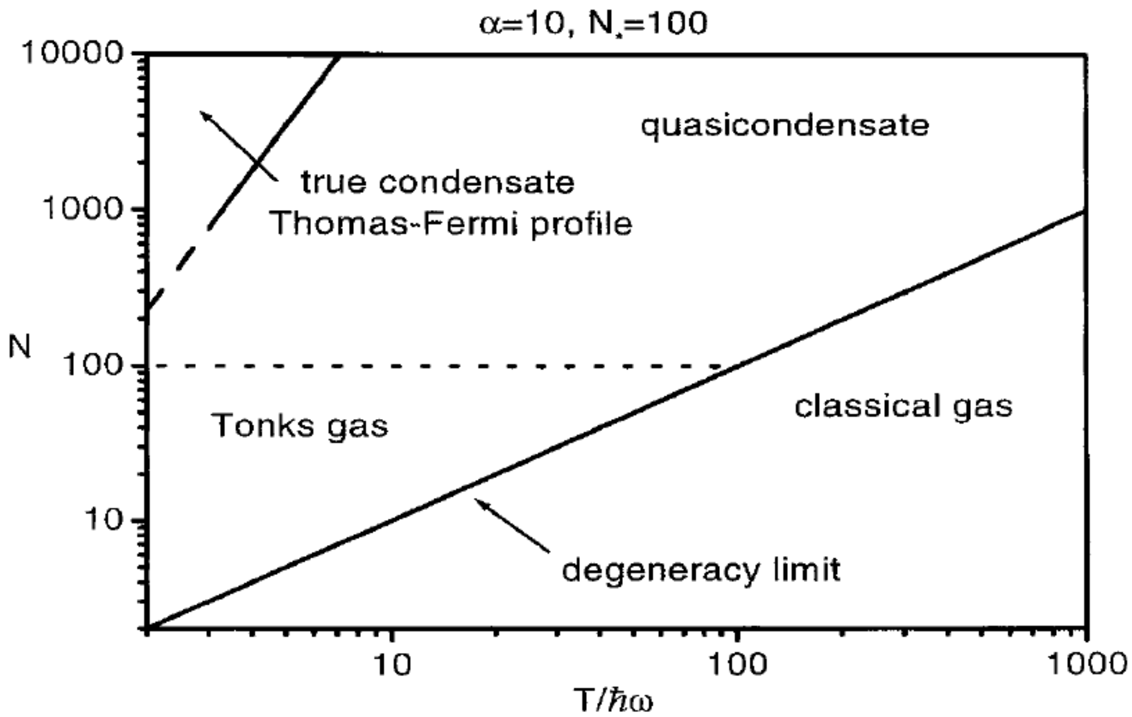


FIG. 1. Diagram of states for a trapped 1D gas.

Regimes of Quantum Degeneracy in Trapped 1D Gases
D. S. Petrov, G. V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. **85**, 3745 (2000)

Gross-Pitaevskii equation

Solution of the GP equation

- The lowest energy solution defines the order parameter and is in general real
- Exited state solutions are usually complex, most famous example is the vortex state
- Since GP is a nonlinear equation, two solutions ψ_a and ψ_b corresponding to μ_a and μ_b are not necessarily orthogonal. (The many body wavefunctions are orthogonal but the order parameter not necessarily.)

GP is similar to the Ginzburg-Landau equations:

1. GL-equation: $$\alpha\Psi + \beta|\Psi|^2\,\Psi + \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)^2\Psi = 0$$

2. GL-equation: $$\vec{j}_s = \frac{q\hbar}{m}\frac{1}{2i}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) - \frac{q^2}{m}|\Psi|^2\,\vec{A}$$

Supplements

Gross-Pitaevskii equation

Conditions for the GP equation:

1. Large number of atoms ($\leftrightarrow$ concept of BEC)

2. Dilute gas $|a| << n^{-1/3}$ and low T
   With these conditions both, the quantum and thermal depletion of the condensate can be neglected and the order parameter can be normalized to the total number of atoms.

3. GP describes only phenomena taking place on length scales >> a

Φ is analogous to E and B of the Maxwell theory. So the condensate wavefunction represents the classical limit of the de Broglie waves, where the corpuscular aspect of matter does not matter.

Difference between GP and Maxwell equations:

1. GP contains the Planck constant so the value of ħ enters into coherence and interference phenomena. The reason is the different dispersion relation.

   Photon: $E = cp \;\rightarrow\; \omega = ck$ Atom: $E = \dfrac{p^2}{2m} \;\rightarrow\; \omega = \dfrac{\hbar k^2}{2m}$

2. GP is nonlinear. This raises similarities to nonlinear optics.

Hydrodynamic equations

Differential equations for density and velocity

The wavefunction $\psi(\vec{r},t)=\sqrt{n(\vec{r},t)}\cdot e^{i\varphi(\vec{r},t)}$

is substituted into the time dependent GP equation

$$i\hbar\dot{\psi}(\vec{r},t)=\left(-\frac{\hbar^2\nabla^2}{2M}+V(\vec{r})+g\left|\psi(\vec{r},t)\right|^2\right)\psi(\vec{r},t)$$

Sorting real and imaginary parts results in two equations:

Equation of continuity: $\dot{n}+\nabla(nv)=0$

Quantum-Bernouli equation:
(force balance)

$$M\dot{v}+\nabla\left(V(\vec{r})+gn-\underbrace{\frac{\hbar^2}{2M\sqrt{n}}\nabla^2\sqrt{n}}_{\text{quantum pressure}}+\frac{Mv^2}{2}\right)=0$$

For usual condensates is the quantum pressure negligible and we get the Euler equation that is known from the frictionless hydrodynamics.

$$M\dot{v}+\nabla\left(V(\vec{r})+gn+\frac{Mv^2}{2}\right)=0$$ The condensate is a superfluid!

Eigenfrequencies of collective excitations for $Na/a_{ho} >> 1$, small amplitude oscillations

The hydrodynamic equations

$$\dot{n} + \nabla(nv) = 0$$

$$M\dot{v} + \nabla\left(V(\vec{r}) + gn - \cancel{\frac{\hbar^2}{2M\sqrt{n}}\nabla^2\sqrt{n}} + \frac{Mv^2}{2}\right) = 0$$

gives for the density modulation (after linearization in δn):

$$\frac{\partial^2}{\partial t^2}\delta n(\mathrm{x},t) = \nabla\left((\mu - U(\mathrm{x}))\nabla\delta n(\mathrm{x},t)\right)$$

For spherical potentials the solutions are of the form

$$\delta n(\mathrm{r}) = P_l^{2n_r}(r/R)r^l Y_{lm}(\theta,\phi)$$

<u>The dispersion law for the discretized normal modes:</u>

$$\omega^2(n_r,l) = \omega_{\mathrm{HO}}^2(2n_r^2 + 2n_r l + 3n_r + l)$$

with ansatz (small perturbations on the top of the stationary solution of the GP equation)

$$\begin{aligned} n_0(\mathrm{x}) &= 1/g(\mu - U(\mathrm{x})) \\ n(\mathrm{x},t) &= n_0(\mathrm{x}) + \delta n(\mathrm{x},t) \\ \mathrm{v}(\mathrm{x},t) &= \delta\mathrm{v}(\mathrm{x},t). \end{aligned}$$

Ansatz: periodic modulation

$$\delta n(\mathrm{x},t) = \delta n(\mathrm{x})e^{i\omega t}$$

P are polinoms of $2n_r$ order, n_r is the radial quantum number, *l* and *m* are quantum numbers of the angular momentum.

Different from the case of the ideal gas!

$$\begin{aligned} \omega_{\mathrm{iG}}(n_r,l) &= \omega_{\mathrm{HO}}(n_x + n_y + n_z) \\ &= \omega_{\mathrm{HO}}(2n_r + l) \end{aligned}$$

Eigenfrequencies of collective excitations for Na/a_{ho}>>1, small amplitude oscillations

For a trap with cylindrical symmetry (explicit results are known for some particular cases):

$$\begin{aligned}\omega^2(l, m = \pm l) &= l\omega_r^2 \\ \omega^2(l, m = \pm(l-1)) &= (l-1)\omega_r^2 + \omega_a^2\end{aligned}$$

The modes (n_r,l,m) = (0,2,0) and (1,0,0) are coupled in the cylindrical trap.

The corresponding new eigenfreuencies are:

$$\omega_\pm^2(m=0) = 2\omega_r^2 + \frac{3}{2}\omega_a^2 \pm \frac{1}{2}\sqrt{9\omega_a^4 - 16\omega_a^2\omega_r^2 + 16\omega_r^4}$$

The lowest lying eigenmodes:

$$\omega_- = \sqrt{\frac{5}{2}}\omega_a$$

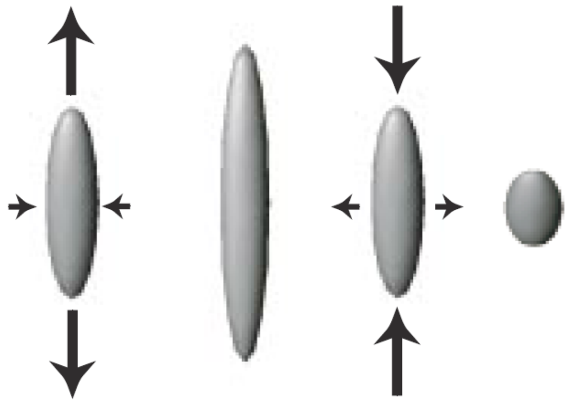
These are observed in numerous experiments.

$$\omega_+ = 2\omega_r$$

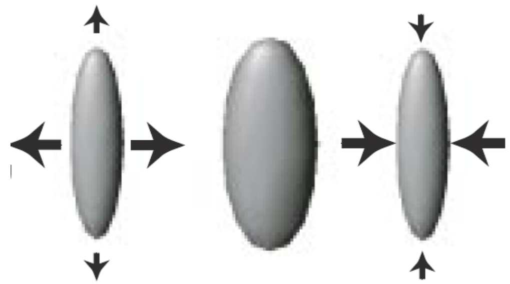
Collective small amplitude excitations

Low frequency, small amplitude shape oscillations of the condensate in a trap.

For a condensate in a trap with cylindrical symmetry, the lowest lying excitations are the following quadrupole modes:



axial shape oscillation
(small radial contribution) $\omega = \sqrt{5/2}\,\omega_a$



radial shape oscillation
(small axial contribution) $\omega = 2\omega_r$

radial shape oscillation $\omega = \sqrt{2}\,\omega_r$

The center of mass oscillation of the condensate in the trap ($\omega=\omega_a$, ω_r) is referred as dipole oscillation.

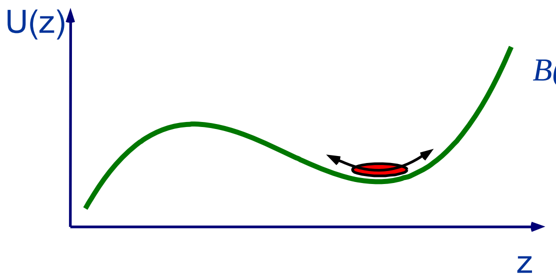
Condensate oscillation in anharmonic traps

initial displacement:

A: 0.45 mm

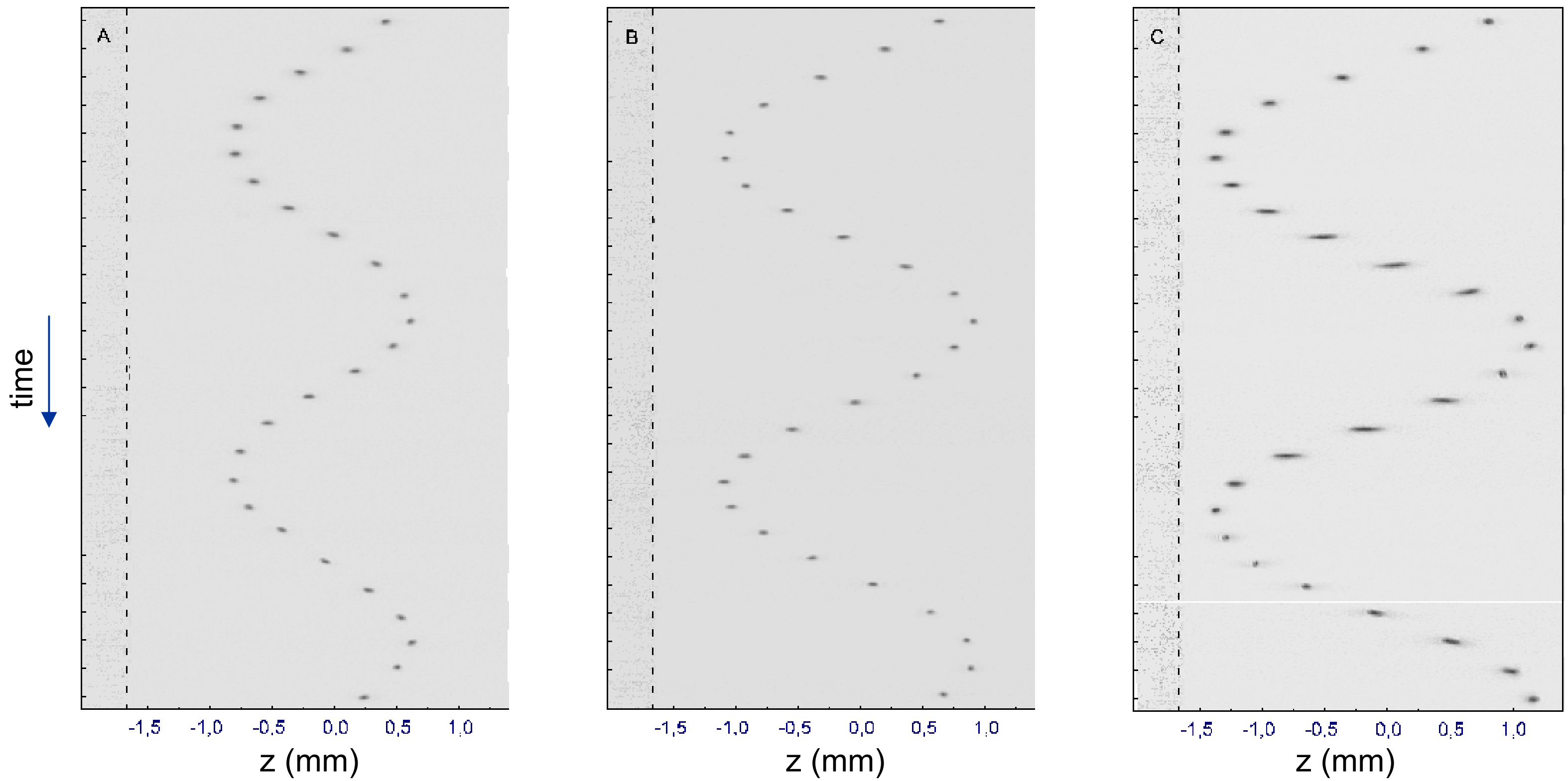
B: 0.63 mm

C: 0.79 mm



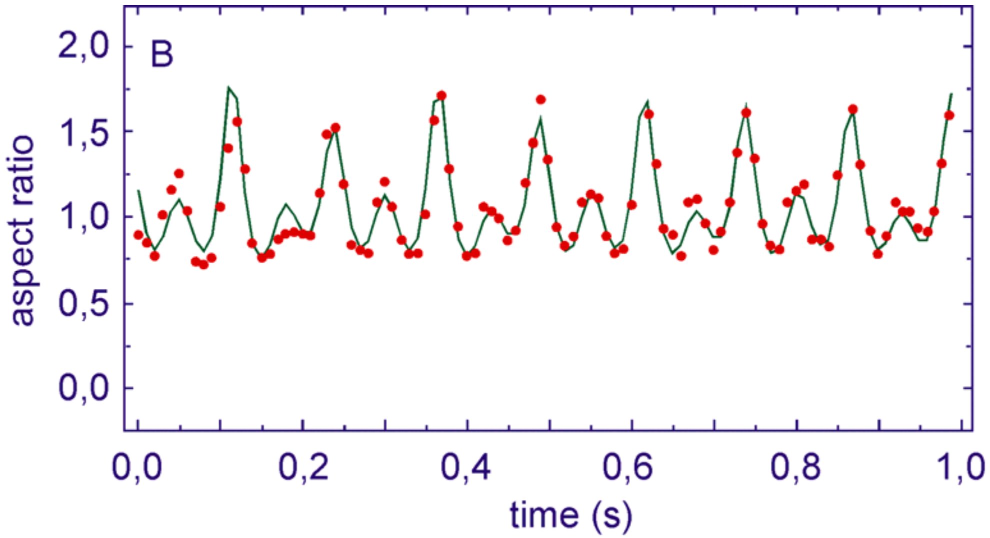
$B(z)$ = 3.879 + 0.219 z^2 - 0.011 z^3 – 0.026 z^4

15 ms time of flight



Shape oscillations

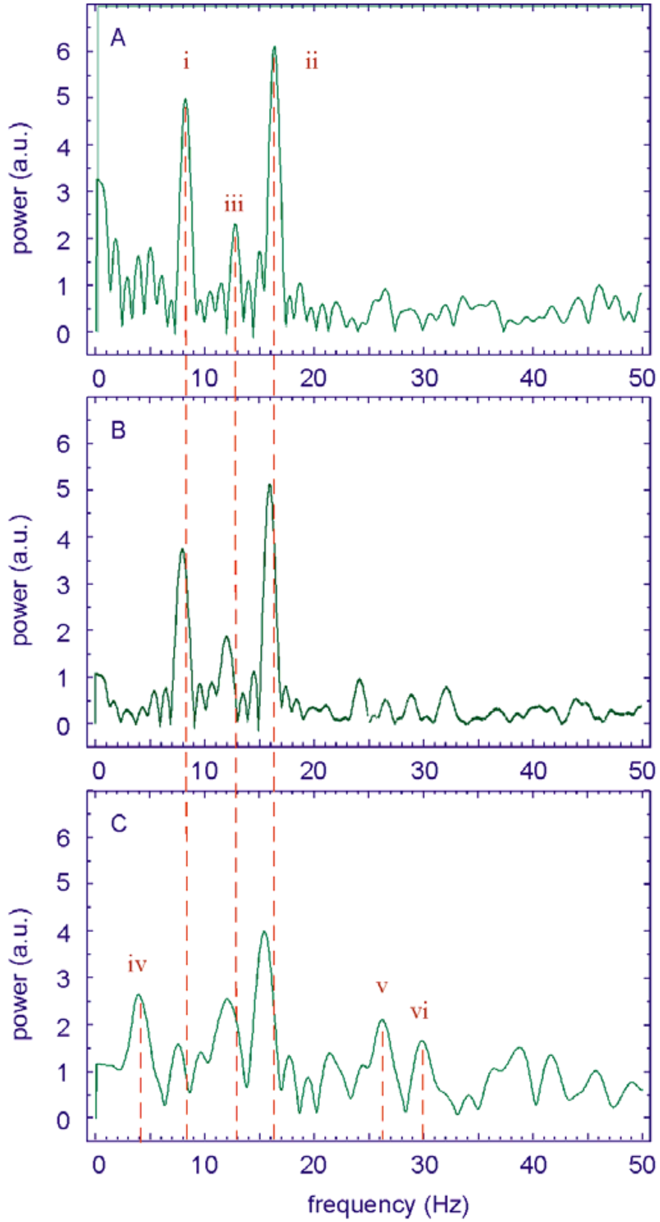
aspect ratio r/z after 15 *ms* time of flight



- data well described by the Gross-Pitaevskii equation
- no damping
- slowly increasing offset due to transition into 1D regime

H. Ott, J. Fortágh, S. Kraft, A. Günther,
D. Komma, C. Zimmermann
PRL 91, 040402 (2003)

Spectra of shape oscillations



- excitation of the fundamental (i)
- excitation of the 2nd harmonic (ii)
- off-resonant excitation of lowest collective mode (iii)
- nonlinear sum frequency mixing between all modes

transition to deterministic chaos

Large amplitude oscillations: scaling equations

We derive the equations for the expansion of a condensate in the Thomas-Fermi regime from a harmonic trap. The theory is given by Castin and Dum PRL 77, 5316 (1996) and describes the evolution of the condensate density and the phase.

The ansatz solves the hydrodynamic equations:

$$n(\vec{r},t) = a_0(t) - a_x(t)x^2 - a_y(t)y^2 - a_z(t)z^2$$

$$v(\vec{r},t) = \frac{1}{2}\nabla\left(\alpha_x(t)x^2 + \alpha_y(t)y^2 + \alpha_z(t)z^2\right)$$

This results 6 coupled differential equations for a_i and α_i. The 7th parameter a_0 is given by the normalization.

A practical scaling is the Thomas-Fermi radius:

$$R_i(0) = \sqrt{\frac{2\mu}{M\omega_{0i}^2}} \longrightarrow R_i(t) = R_i(0)\cdot b_i(t)$$

scaling parameter
describing the size of the condensate at any time in units of the Thomas-Fermi radius

Inserting into the differential equations $\quad a_i = M\omega_{i0}^2 \frac{1}{2gb_xb_yb_zb_i^2}$

results the scaling equations: $\quad \dot{b}_i = \alpha_i b_i \quad$ and $\quad \ddot{b}_i + \omega_i^2(t)b_i + \frac{\omega_{0i}^2}{b_xb_yb_zb_i} = 0$

Scaling equations, expansion

Starting with a condensate in a trap, all velocities are 0 $\longrightarrow$ $\alpha_l = 0$,

the phase over the condensate is constant,

b_i=1 and b_i/dt=0.

The scaling equations say that the condensate keeps a parabolic density distribution when the trap frequencies change.

The simplest case: sudden turn off the trap

$$\ddot{b}_i - \frac{\omega_{0i}^2}{b_x b_y b_z b_i} = 0$$

For a trap with cylindrical symmetry:

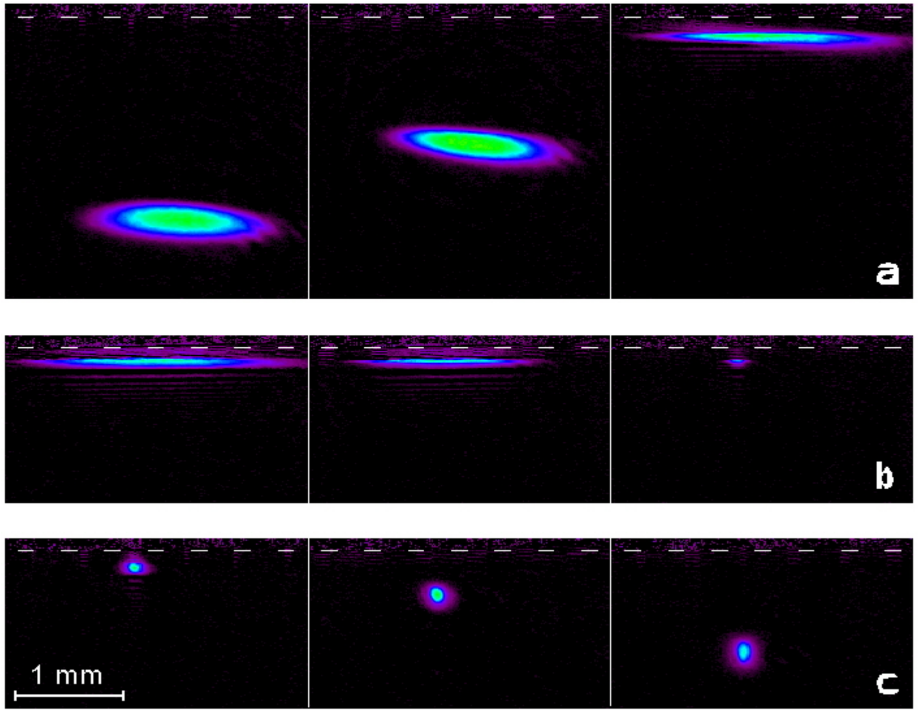
$$\begin{aligned} \omega_\perp &: = \omega_y = \omega_x \\ \omega_{||} &: = \omega_z \end{aligned} \longrightarrow \omega_{||} = \lambda \omega_\perp$$

For a cigar shaped condensate λ<<1, we got analytic solutions

$$\begin{aligned} b_\perp &= \sqrt{1+\tau^2} \\ b_{||} &= 1 + \lambda^2 \left(\tau \arctan(\tau) - \ln \sqrt{1+\tau^2} \right) \end{aligned} \qquad \tau := t \cdot \omega_\perp$$

The phase of the condensate takes a parabolic distribution and its evolution is given also by the scaling parameter (see original paper).

Scaling equations, expansion



scaling parameters

describing the radial and axial size of the condensate at any time in units of the Thomas-Fermi radius

$$b_\perp = \sqrt{1+\tau^2}$$
$$b_{\shortparallel} = 1 + \lambda^2 \left(\tau \arctan(\tau) - \ln\sqrt{1+\tau^2}\right)$$

$$\tau := t \cdot \omega_\perp$$

The phase of the condensate takes a parabolic profile and its evolution is given also by the scaling parameter (see original paper).